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Variation of Moments With Distributions of Masses and Areas

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VARIATION OF MOMENTS

WITH DISTRIBUTIONS OF MASSES AND AREAS

being

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by

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INTRODUCTION

The purpose of this investigation has been to determine the variations of the first, second, third and fourth moments as functions of distributions of masses and areas. These moments were found about a line or plane perpendicular to the horizontal axes of the figures, and at various distances along the axes. For a complete understanding of the problem, an explanation of various terms will be made in relation to mechanics and statistics.

In mechanics the first moment of a force, or torque, is known to be the product of a force and its lever arm. The second, known as the moment of inertia, is the product of the force and the square of the lever arm. For higher moments the lever arm is raised to the power corresponding to the moment and multiplied by the force. The lever arm of a force is the perpendicular distance from the line of action of the force to the axis of rotation. The radius of gyration of a body whose moment of inertia is I and whose mass is M is defined by the quantity $\sqrt{I/M}$. The radius of gyration equals the distance from the axis at which a particle of mass M must be placed in order to have the same moment of inertia as the original mass.

The center of gravity of a body is that point about which the algebraic sum of the first moments is zero. In other words, the center of gravity of a body acted upon by gravity is a point, such that, no matter how the body is placed, it will not tend to rotate about a fixed horizontal axis through the point. The center of gravity, the center of mass, the center of inertia and the centroid (assuming gravitational lines parallel) are one and the same point, but the center of matter is not necessarily the same. If a body has a geometrical center, that point is the centroid and any plane or line of symmetry must contain the centroid.

Statistically the first moment about a perpendicular line through the horizontal axis is the frequency multiplied by the distance from the line. The second order momemt in statistics is the frequency multiplied by the square of the lever arm or distance. Likewise higher moments are found by raising the lever arm to the power of the moment and multiplying by the frequency. "A frequency distribution is an arrangement which shows the frequencies of the values of a variable in ordered classes."[1]

The first step in the procedure for developing this thesis was the setting up of integrals within definite limits. These integrals represent the first, second, third, and fourth moments taken with respect to lines or planes at regular intervals through the following bodies: homogeneous and heterogeneous bars, a circle, an ellipse, a parabola, and a cone. The integrals were solved and the resulting moments, as a function of the distance along the body, were plotted on graph paper to show the nature of the curves formed. The results for the various bodies were compared and generalizations were made.

For statistics the first moment about the origin gives the mean

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1. Rietz, H. L. Handbook of Mathematical Statistics, p. 20.

value. However, the most important moment about the origin is the second moment. The square root of the second moment about the arithmetic mean is called the standard deviation. The third and fourth moments are usually expressed in terms of the standard deviation. In general, the second moment is a measure of dispersion. It shows how widely the data is spread out on either side of the mean. The standard deviation is the physicists "radius of gyration." From the third moment one can determine the "skewness," the bulging of a distribution on one side more than the other. From the fourth moment is found the "kurtosis." A distribution has kurtosis if the material is spread out on either side to a much greater distance than the extent of the standard deviation.

The author of this thesis wishes to acknowledge all sources of information used in its preparation. He is especially indebted to Dr. C. T. McCormick, professor in charge of thesis, for the valuable assistance which he gave so kindly during the writing of this treatise. He also wishes to express thanks to the following members of the faculty of Fort Hays Kansas State College for helpful suggestions and materials: Professor E. E. Colyer, Dr. F. B. Streeter, Dr. George A. Kelly, and Dr. H. A. Zinszer.

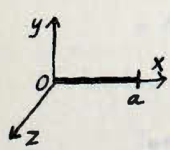
CHAPTER I

Variation of First Moments

The demonstration of moments can be most easily understood by beginning with the first moment. The effect of this moment is to produce rotation around the axis with respect to which the moment is taken.

> The product of the mass m, concentrated at a point P, by the distance <u>x</u> of P from a given point, line or plane, is called the mass-moment, simple rotation or moment of the first order of <u>m</u> with respect to the point, line or plane. Denoting this moment by G, we have
>
> $$G = mx$$
>
> If a system of points $P_1, P_2, \dots, P_n$, having masses $m_1, m_2, \dots, m_n$, respectively, be referred to Cartesian coordinate axes, the moments of the system with respect to the three coordinate planes are respectively
>
> $$G_{yz} = \sum_{i=1}^{n} m_i x_i, \quad G_{zx} = \sum_{i=1}^{n} m_i y_i, \quad G_{xy} = \sum_{i=1}^{n} m_i z_i.$$ [1]

To begin with a very simple illustration, the first moments of a homogeneous bar, (Fig. 1 and 8), of constant density <u>k</u> and of length <u>a</u> will be found. The moments are found about lines perpendicular to the <u>x</u> axis and at various intervals along the axis.



Let $\bar{x}$ = the x - Coordinate of the centroid.

G_{yz} = moment of mass with respect to the YOZ plane = $M\bar{x}$

Where M = mass of the uniform bar

Fig. 1

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1. Love. <u>Differential and Integral Calculus</u>, third edition, p. 204.

$M = ka, \quad dm = k\,dx, \quad m = k\,(\Delta x)$

$M\bar{x} = \int x\,dm$

$M\bar{x} = x_1(\Delta m_1) + x_2(\Delta m_2) + x_3(\Delta m_3) + \ldots + x_n(\Delta m_n)$ approximately true.

$= x_1(k\Delta x_1) + x_2(k\Delta x_2) + x_3(k\Delta x_3) + \ldots + x_n(k\Delta x_n)$ approximately true.

With n fairly large, this is approximately true if elements of mass are not equal and also if the lengths of x, i.e. x_1, x_2, x_3, etc. are taken anywhere in the element (not necessarily at midpoint or center) but as $n \to \infty$ and $x \to o$ this represents exactly $M\bar{x}$, that is

$$M\bar{x} = \lim_{n\to\infty} \sum_{1}^{n} kx_1(\Delta x_1) + kx_2(\Delta x_2) + kx_3(\Delta x_3) + \ldots + kx_n(\Delta x_n)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} kx_i(\Delta x_i) = k\int_0^a x\,dx = \left[kx^2/2\right]_0^a$$

$$= ka^2/2 = 1/2(ka)\,a = 1/2\,Ma$$

$\bar{x} = a/2$, which is true for a uniform bar with any shaped cross sectional area.

The first moment, I_y, about the origin may be determined by

$$I_y = \int_0^a kx\,dx = \left[kx^2/2\right]_0^a = ka^2/2 = Ma/2$$

The first moment about a line perpendicular to the axis and through the point a/8 could be determined by

$$I_y = \int_0^{7a/8} kx\,dx + \int_0^{a/8} k(x-a/8)dx = \left[kx^2/2\right]_0^{7a/8} + \left[kx^2/2 - kax/8\right]_0^{a/8}$$

$$= 49ka^2/128 + ka^2/128 - ka^2/64 = 48ka^2/128 = 3Ma/8$$

A simpler notation that gives the same result is

$$I_y = \int_0^a k\,(x-a/8)\,dx = \left[kx^2/2 - kax/8\right]_0^a = ka^2/2 - ka^2/8$$

$$= 3ka^2/8 = 3Ma/8$$

The first moment about a line through the point a/4 is

$$I_y = \int_0^a k\ (x-a/4)\ dx = \left[kx^2/2 - kax/4\right]_0^a$$

$$= ka^2/2 - ka^2/4 = ka^2/4 = Ma/4$$

The moment about a line through 3a/8 is

$$I_y = \int_0^a k\ (x-3a/8)\ dx = \left[kx^2/2 - 3kax/8\right]_0^a$$

$$= ka^2/2 - 3ka^2/8 = ka^2/8 = Ma/8$$

The moment about a line through a/2, the center of mass, is 0.

$$I_y = \int_0^a k(x-a/2)\ dx = \left[kx^2/2 - kax/2\right]_0^a = ka^2/2 - ka^2/2 = 0$$

The moment about a line through 5a/8 is

$$I_y = \int_0^a k(x-5a/8)\ dx = \left[kx^2/2 - 5kax/8\right]_0^a$$

$$= ka^2/2 - 5ka^2/8 = -Ma/8$$

The moment about a line through 3a/4 is

$$I_y = \int_0^a k(x-3a/4)\ dx = \left[kx^2/2 - 3kax/4\right]_0^a$$

$$= ka^2/2 - 3ka^2/4 = -Ma/4$$

The moment about a line through 7a/8 is

$$I_y = \int_0^a k(x-7a/8)\ dx = \left[kx^2/2 - 7kax/8\right]_0^a$$

$$= ka^2/2 - 7ka^2/8 = -3Ma/8$$

and the moment about a line through a could be considered as

$$I_y = \int_0^a k(x-a)\ dx = \left[kx^2/2 - kax\right]_0^a$$

$$= ka^2/2 - ka^2 = -Ma/2$$

A graph of these results may be found at the end of this chapter. When plotted, the results determine a straight line.

A slightly more complicated problem may be obtained if we attempt to determine the moments for a heterogeneous bar, (Fig. 2 and 9), whose mass varies as the distance from one end, indicated by point o in Fig. 2.

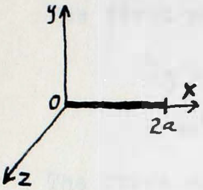


Fig. 2

Let 2a = length and σ = density

$\sigma = kx = \sigma_{av}\,(2a)$ $\qquad dM = kx\,dx$

So $M = ka\,(2a) = 2ka^2$

or $M = \int_0^{2a} \sigma\,dx = \int_0^{2a} kx\,dx = 2ka^2$

The first moment about the origin would be

$$I_y = \int_0^{2a} x(kx)dx = \left[1/3kx^3\right]_0^{2a} = 8/3\ ka^3 = 4Ma/3$$

The first moment about a line through a/4 is

$$I_y = \int_0^{2a} (x - a/4)kx\,dx = \left[kx^3/3 - kax^2/8\right]_0^{2a}$$
$$= 8ka^3/3 - ka^3/2 = 13\ Ma/12$$

The first moment about a line through a/2 is

$$I_y = \int_0^{2a} (x - a/2)kx\,dx = \left[kx^3/3 - kax^2/4\right]_0^{2a}$$
$$= 8ka^3/3 - ka^3 = 5\ Ma/6$$

The first moment about a line through 3a/4 is

$$I_y = \int_0^{2a} (x - 3a/4)kx\,dx = \left[kx^3/3 - 3kax^2/8\right]_0^{2a}$$
$$= ka^3\ (8/3 - 3/2) = 7\ Ma/12$$

The first moment about a line through <u>a</u> is

$$I_y = \int_0^{2a} (x - a)kx\,dx = \left[kx^3/3 - kax^2/2\right]_0^{2a}$$
$$= ka^3(8/3 - 2) = Ma/3$$

The first moment about a line through 5a/4 is

$$I_y = \int_0^{2a} (x - 5a/4)kx\,dx = \left[kx^3/3 - 5kax^2/8\right]_0^{2a}$$
$$= ka^3\ (8/3 - 5/2) = Ma/12$$

The first moment about a line through 3a/2 is

$$I_y = \int_0^{2a} (x - 3a/2)kx\,dx = \left[kx^3/3 - 3kax^2/4\right]_0^{2a}$$
$$= kx^3\ (8/3 - 3) = -\ Ma/6$$

The first moment about a line through 7a/4 is

$$I_y = \int_0^{2a} (x - 7a/4)\, kx\, dx = \Big[kx^3/3 - 7kax^2/8\Big]_0^{2a}$$

$$= ka^3\ (8/3 - 7/2) = -5\ Ma/12$$

The first moment about a line through 2a is

$$I_y = \int_0^{2a} (x - 2a)\ kx\ dx = \Big[kx^3/3 - kax^2\Big]_0^{2a}$$

$$= ka^3(8/3 - 4) = -2\ Ma/3$$

In a similar manner the moments for a heterogeneous bar, (Fig. 3 and 10), whose mass varies as the square of the distance from one end may be determined.

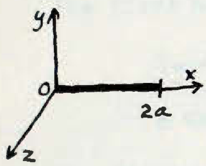


Fig. 3

Let 2a = length and ϕ = density

$$\phi = kx^2 \qquad\qquad dM = kx^2\ dx$$

$$M = \int_0^{2a} kx^2\ dx = \Big[kx^3/3\Big]_0^{2a} = 8\ ka^3/3$$

The first moment about the origin would be

$$I_y = \int_0^{2a} x(kx^2\ dx) = \Big[k\ x^4/4\Big]_0^{2a} = 4\ ka^4 = 3\ Ma/2$$

The first moment about a line through a/4 is

$$I_y = \int_0^{2a} (x - a/4)kx^2\ dx = \Big[kx^4/4 - kax^3/12\Big]_0^{2a}$$

$$= 4ka^4 - 2ka^4/3 = 10/3\ ka^4 = 5\ Ma/4$$

The first moment about a line through a/2 is

$$I_y = \int_0^{2a} (x - a/2)kx^2\ dx = \Big[kx^4/4 - kax^3/6\Big]_0^{2a}$$

$$= 4ka^4 - 4\ ka^4/3 = 8\ ka^4/3 = Ma$$

The first moment about a line through 3a/4 is

$$I_y = \int_0^{2a} (x - 3a/4)kx^2\ dx = \Big[kx^4/4 - 3kax^3/12\Big]_0^{2a}$$

$$= 4ka^4 - 2ka^4 = 2ka^4 = 3\ Ma/4$$

The first moment about a line through $\underline{a}$ is

$$Iy = \int_0^{2a} (x - a)kx^2 dx = \left[kx^4/4 - kax^3/3\right]_0^{2a}$$

$$= 4ka^4 - 8\ ka^4/3 = 4\ ka^4/3 = Ma/2$$

The first moment about a line through 5a/4 is

$$Iy = \int_0^{2a} (x - 5a/4)kx^2\ dx = \left[kx^4/4 - 5kax^3/12\right]_0^{2a}$$

$$= 4ka^4 - 10/3\ ka^4 = 2\ ka^4/3 = Ma/4$$

The first moment about a line through 3a/2 is

$$Iy = \int_0^{2a} (x - 3a/2)kx^2\ dx = \left[kx^4/4 - kax^3/2\right]_0^{2a}$$

$$= 4ka^4 - 4ka^4 = 0$$

The first moment about a line through 7a/4 is

$$Iy = \int_0^{2a} (x - 7a/4)kx^2\ dx = \left[kx^4/4 - 7kax^3/12\right]_0^{2a}$$

$$= 4ka^4 - 14/3\ ka^4 = -\ 2\ ka^4/3 = -\ Ma/4$$

The first moment about a line through 2a is

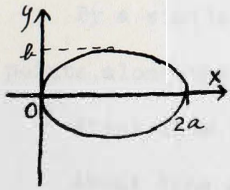
$$Iy = \int_0^{2a} (x - 2a)kx^2\ dx = \left[kx^4/4 - 2kax^3/3\right]_0^{2a}$$

$$= 4ka^4 - 16/3\ ka^4 = -\ 4\ ka^4/3 = -\ Ma/2$$

A similar problem is that of finding the variation of the moments of area of an ellipse, (Fig. 4 and 11), about lines perpendicular to the axis and at various intervals along the axis.

Let the general equation for an ellipse be

$$(x - a)^2/a^2 + y^2/b^2 = 1$$



Then $y = \pm\ b/a\ \sqrt{2ax - x^2}$

$$A = 2\int_0^{2a} b/a\ \sqrt{2ax - x^2}\ dx$$

$$= 2\left\{b/a\ \left[(x - a)/a\right] \sqrt{2ax - x^2} + b/a\ (a^2/2)\ \sin^{-1}\left[(x - a)/a\right]\right\}$$

$$= ab\ (\pi/2 + \pi/2) = ab\pi$$

Fig. 4

The first moment of area about the y axis is

$$I_y = \int_0^{2a} x \cdot b/a \sqrt{2ax - x^2}\, dx = \left[- 2b/3a \sqrt{(2ax - x^2)^3}\right]_0^{2a} + 2b \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \left[- 2b/3a \sqrt{(2ax - x^2)^3}\right]_0^{2a} + \left[2b\,(x - a)/2 \sqrt{2ax - x^2} + a^2/2 \sin^{-1} (x - a)/a\right]_0^{2a}$$

$$= 0 + 2b\,(a^2 \pi/4 - a^2 \pi/4) = ba^2\pi = Aa$$

The first moment of area about the line a/8 is

$$I_y = 2 \int_0^{2a} (x - a/8)\, b/a \sqrt{2ax - x^2}\, dx = 2b/a \int_0^{2a} x \sqrt{2ax - x^2}\, dx - b/4 \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \left[- 2b/3a \sqrt{(2ax - x^2)^3}\right]_0^{2a} + 2b \int_0^{2a} \sqrt{2ax - x^2}\, dx - b/4 \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \left[- 2b/3a \sqrt{(2ax - x^2)^3} + 7b/4 \cdot (x - a)/2 \sqrt{2ax - x^2} + a^2/2 \sin^{-1} (x - a)/a\right]_0^{2a}$$

$$= 0 - 7b/4 \left[a^2/2 \cdot \pi/2 - a^2/2\,(- \pi/2)\right] = 7ba^2\pi/8 = 7/8\, Aa$$

The first moment of area about the line a/4 is

$$I_y = 2 \int_0^{2a} (x - a/4)\, b/a \sqrt{2ax - x^2}\, dx = 2b/a \int_0^{2a} x \sqrt{2ax - x^2}\, dx - b/2 \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \left[- 2b/3a \sqrt{(2ax - x^2)^3}\right]_0^{2a} + 2b \int_0^{2a} \sqrt{2ax - x^2}\, dx - b/2 \int_0^{2a} \sqrt{2ax - x^2}\, dx$$

$$= \left[- 2b/3a \sqrt{(2ax - x^2)^3}\right]_0^{2a} + \left[3b/2 \cdot (x - a)/2 \cdot \sqrt{2ax - x^2} - a^2/2 \sin^{-1} (x - a)/a\right]_0^{2a}$$

$$= 0 + 3b/2 \left[a^2/2 \cdot \pi/2 - a^2/2\,(- \pi/2)\right] = 3ba^2\pi/4 = 3/4\, Aa$$

By a similar method the moments may be found about lines through other points along the axis

About line 3a/8	$I_y = 5/8$ Aa
About line a/2	$I_y = 1/2$ Aa
About line 5a/8	$I_y = 3/8$ Aa
About line 3a/4	$I_y = 1/4$ Aa

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About line 7a/8 $I_y = 1/8\ Aa$

About line a $I_y = 0$

About line 9a/8 $I_y = -\ 1/8\ Aa$

About line 5a/4 $I_y = -\ 1/4\ Aa$

About line 11a/8 $I_y = -\ 3/8\ Aa$

About line 3a/2 $I_y = -\ 1/2\ Aa$

About line 13a/8 $I_y = -\ 5/8\ Aa$

About line 7a/4 $I_y = -\ 3/4\ Aa$

About line 15a/8 $I_y = -\ 7/8\ Aa$

About line 2a $I_y = -\ Aa$

A similar problem which gives identical results is that of finding the moments of the area of a circle, (Fig. 5 and 12), a special form of the ellipse, about lines perpendicular to the axis.

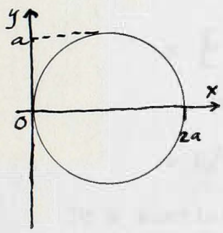


Fig. 5

Let the equation of the circle be $(x - a)^2 + y^2 = a^2$

Then $y = \pm\sqrt{a^2 - (x - a)^2} = \pm\sqrt{2ax - x^2}$

If A = area

$$A = 2\int_0^{2a} y\ dx = 2\int_0^{2a} \sqrt{2ax - x^2}\ dx$$

$$= 2\ \left[(x - a)/2\sqrt{2ax - x^2} - a^2/2\ \sin^{-1}\ (x - a)/a\right]_0^{2a}$$

$$= a^2\ \left[\pi/2 - (-\ \pi/2)\right] = \pi a^2$$

The first moment of area about the y axis is

$$I_y = 2\int_0^{2a} x\sqrt{2ax - x^2}\ dx = -\ 2/3\ \sqrt{(2ax - x^2)^3} + 2a\int_0^{2a} \sqrt{2ax - x^2}\ dx$$

$$= \Big[-\ 2/3\ \sqrt{(2ax - x^2)^3} + 2a\ (x - a)/2\ \sqrt{2ax - x^2} + a^3\ \sin^{-1}\ (x - a)/a\Big]_0^{2a}$$

$$= a^3\ \left[\pi/2 - (-\ \pi/2)\right] = a^3\ \pi = Aa$$

The first moment about a line through $a/8$ is

$$I_y = 2\int_0^{2a} (x - a/8)\sqrt{2ax - x^2}\,dx = -2/3\sqrt{(2ax - x^2)^3} +$$

$$2a\int_0^{2a}\sqrt{2ax - x^2}\,dx - a/4\int_0^{2a}\sqrt{2ax - x^2}\,dx$$

$$= \Big[-2/3\sqrt{(2ax - x^2)^3} + 7a/4\,(x - a)/2\,\sqrt{2ax - x^2} - 7a/4\,(a^2/2)\cdot$$

$$\sin^{-1}(x - a)/a\Big]_0^{2a}$$

$$= 7/8\,a^3\,[\pi/2 - (-\pi/2)] = 7/8\,Aa$$

The first moment about a line through $a/4$ is

$$I_y = 2\int_0^{2a} (x - a/4)\sqrt{2ax - x^2}\,dx = -2/3\sqrt{(2ax - x^2)^3}$$

$$+ 2a\int_0^{2a}\sqrt{2ax - x^2}\,dx - a/2\int_0^{2a}\sqrt{2ax - x^2}\,dx$$

$$= \Big[-2/3\sqrt{(2ax - x^2)^3} + 3a/2\,(x - a)/2\,\sqrt{2ax - x^2} + 3a/2\,(a^2/2)\cdot$$

$$\sin^{-1}(x - a)/a\Big]_0^{2a}$$

$$= 3/4\,a^3\,[\pi/2 - (-\pi/2)] = 3/4\,a^3\pi = 3/4\,Aa$$

The first moment about a line through $3a/8$ is

$$I_y = 2\int_0^{2a} (x - 3a/8)\sqrt{2ax - x^2}\,dx = -2/3\sqrt{(2ax - x^2)^3} +$$

$$2a\int_0^{2a}\sqrt{2ax - x^2}\,dx - 3a/4\int_0^{2a}\sqrt{2ax - x^2}\,dx$$

$$= \Big[-2/3\sqrt{(2ax - x^2)^3} + 5a/4\,(x - a)/2\,\sqrt{2ax - x^2} + 5a/4\cdot$$

$$(a^2/2)\,\sin^{-1}(x - a)/a\Big]_0^{2a}$$

$$= 5/8\,a^3\,[\pi/2 - (-\pi/2)] = 5/8\,a^3\pi = 5/8\,Aa$$

In a similar manner the following moments may be determined:

About line $a/2$	$I_y = 1/2\,Aa$
About line $5a/8$	$I_y = 3/8\,Aa$
About line $3a/4$	$I_y = 1/4\,Aa$
About line $7a/8$	$I_y = 1/8\,Aa$
About line a	$I_y = 0$
About line $9a/8$	$I_y = -1/8\,Aa$

About line $5a/4$ $\qquad$ $I_y = -1/4\ Aa$

About line $11a/8$ $\qquad$ $I_y = -3/8\ Aa$

About line $3a/2$ $\qquad$ $I_y = -1/2\ Aa$

About line $13a/8$ $\qquad$ $I_y = -5/8\ Aa$

About line $7a/4$ $\qquad$ $I_y = -3/4\ Aa$

About line $15a/8$ $\qquad$ $I_y = -7/8\ Aa$

About line $2a$ $\qquad$ $I_y = -Aa$

To take a figure such as a parabola, (Fig. 6 and 13), which is not symmetrical about the ce ter of gravity the moments of area about lines perpendicular to the axis of the figure will be found to differ slightly from those of the ellipse and circle. Only the area between the limits 0 and a will be considered.

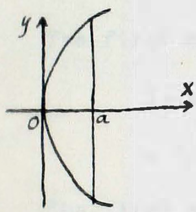


Fig. 6

Let the equation of the parabola be $y^2 = 4ax$

Then $y = \pm 2\sqrt{ax}$

$$A = 2\int_0^a y\,dx = 4\sqrt{a}\int_0^a \sqrt{x}\,dx$$

$$= 4\sqrt{a}\cdot 2/3\,\left[x^{3/2}\right]_0^a = 8/3\ a$$

The first moment about the origin is

$$I_y = 2\int_0^a xy\,dx = 4\sqrt{a}\int_0^a x^{3/2}\,dx$$

$$= 8/5\ a^3 = 3/5\ Aa$$

The first moment about the line $a/8$ is

$$I_y = 2\int_0^a (x - a/8)\,2\sqrt{ax}\,dx = 4\sqrt{a}\int_0^a x^{3/2}\,dx - a^{3/2}\,/2\int_0^a x^{1/2}\,dx$$

$$= \left[(8\sqrt{a}/5)\,x^{5/2} - (a^{3/2}/3)\,x^{3/2}\right] = 8/5\ a^3 - a^3/3 = 19/15\ a^3 = 19/40\ Aa$$

The first moment about the line $a/4$ is

$$I_y = 2\int_0^a (x - a/4)\,2\sqrt{ax}\,dx = \left[(8\sqrt{a}/5)\,x^{5/2} - (2\,a^{3/2}/3)\,x^{3/2}\right]_0^a$$

$$= 8/5\ a^3 - 2/3\ a^3 = 14/15\ a^3 = 7/20\ Aa$$

The first moment about the line 3a/8 is

$$I_y = 2\int_0^a (x - 3a/8)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - a^{3/2}\, x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - a^3 = 3/5\, a^3 = 9/40\, Aa$$

The first moment about the line a/2 is

$$I_y = 2\int_0^a (x - a/2)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - 4/3\, a^{3/2}\, x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - 4/3\, a^3 = 4/15\, a^3 = 1/10\, Aa$$

The first moment about the line 5a/8 is

$$I_y = 2\int_0^a (x - 5a/8)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - 5/3\, a^{3/2}\, x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - 5/3\, a^3 = -\, a^3/15 = -\, 1/40\, Aa$$

The first moment about the line 3a/4 is

$$I_y = 2\int_0^a (x - 3a/4)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - 2\, a^{3/2} x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - 2a^3 = -\, 2/5\, a^3 = 3/20\, Aa$$

The first moment about the line 7a/8 is

$$I_y = 2\int_0^a (x - 7a/8)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - 7/3\, a^{3/2}\, x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - 7/3\, a^3 = -\, 11/15\, a^3 = -\, 11/40\, Aa$$

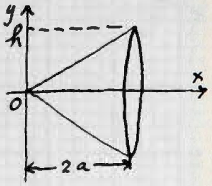
The first moment about the line <u>a</u> is

$$I_y = 2\int_0^a (x - a)\, 2\sqrt{ax}\, dx = \left[(8\sqrt{a}/5)\, x^{5/2} - 8/3\, a^{3/2}\, x^{3/2}\right]_0^a$$

$$= 8/5\, a^3 - 8/3\, a^3 = -\, 16/15\, a^3 = -\, 2/5\, Aa$$

A slightly more complicated problem is that of finding the variation of the moments of the mass of the cone, (Fig. 7 and 14), about planes perpendicular to the axis and parallel to the base of the cone.

In the figure y equals radius of the cone and 2a equals altitude of the cone.



$$y/h = x/2a \quad \text{or} \quad y = hx/2a$$

$$\text{and } y^2 = h^2x^2/4a^2$$

$$M = \int_0^{2a} \pi y^2 dx = \pi h^2/4a^2 \int x^2 \, dx = \left[\pi h^2/12a^2 \; x^3\right]_0^{2a}$$

$$= 8\pi h^2 a^3/12a^2 = 2/3 \; \pi a \; h^2$$

Fig. 7

The first moment of the mass about a plane through the origin or vertex of the cone is

$$I_y = \int_0^{2a} x \pi y^2 \, dx = \pi h^2/4a^2 \int x^3 \, dx = \left[\pi h^2/16a^2 \; x^4\right]_0^{2a}$$

$$= \pi h^2 \; a^2 = 3/2 \; Ma$$

The first moment about a plane through a/4 is

$$I_y = \int_0^{2a} (x - a/4) \pi y^2 \, dx = \pi h^2/4a^2 \int x^3 \, dx - \pi h^2/16a \int x^2 \, dx$$

$$= \left[\pi h^2 x^4/16a^2 - \pi h^2 x^3/48a\right]_0^{2a} = \pi h^2 a^2 - \pi h^2 a^2/6$$

$$= 5/6 \; \pi h^2 a^2 = 5/4 \; Ma$$

The first moment about a plane through a/2 is

$$I_y = \int_0^{2a} (x - a/2) \pi y^2 \, dx = \pi h^2/4a^2 \int x^3 \, dx - \pi h^2/8a \int x^2 \, dx$$

$$= \left[\pi h^2 \; x^4/16 \; a^2 - \pi h^2 x^3/24a\right]_0^{2a} = \pi h^2 a^2 - \pi h^2 a^2/3$$

$$= 2/3 \; \pi \, h^2 a^2 = Ma$$

By similar integration the moments may be found about planes through other points along the axis.

About 3a/4	$I_y = 3/4$ Ma
About a	$I_y = 1/2$ Ma
About 5a/4	$I_y = 1/4$ Ma
About 3a/2	$I_y = 0$
About 7a/4	$I_y = - 1/4$ Ma
About 2a	$I_y = - 1/2$ Ma

The following graphs illustrate the results that have been obtained:

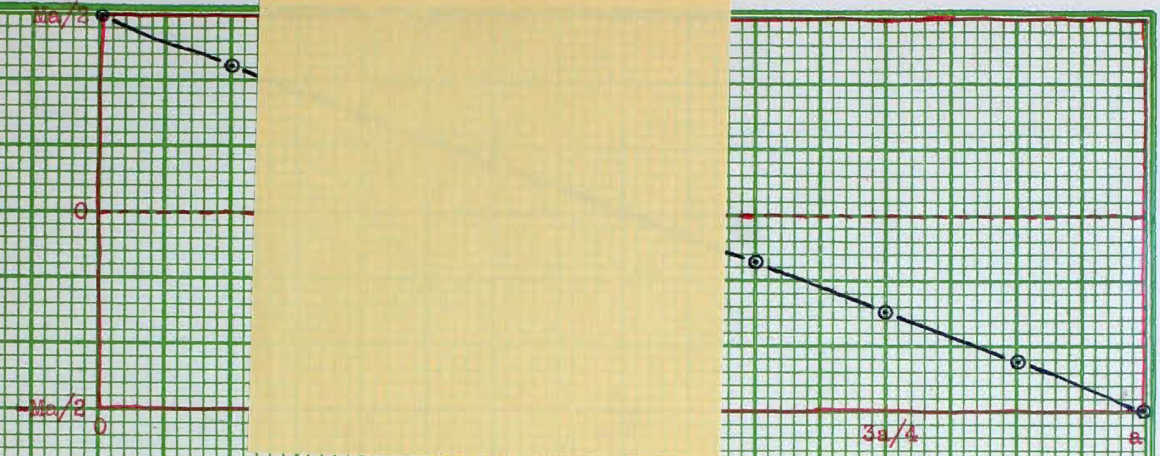


Fig. 8 The Homogeneous Bar

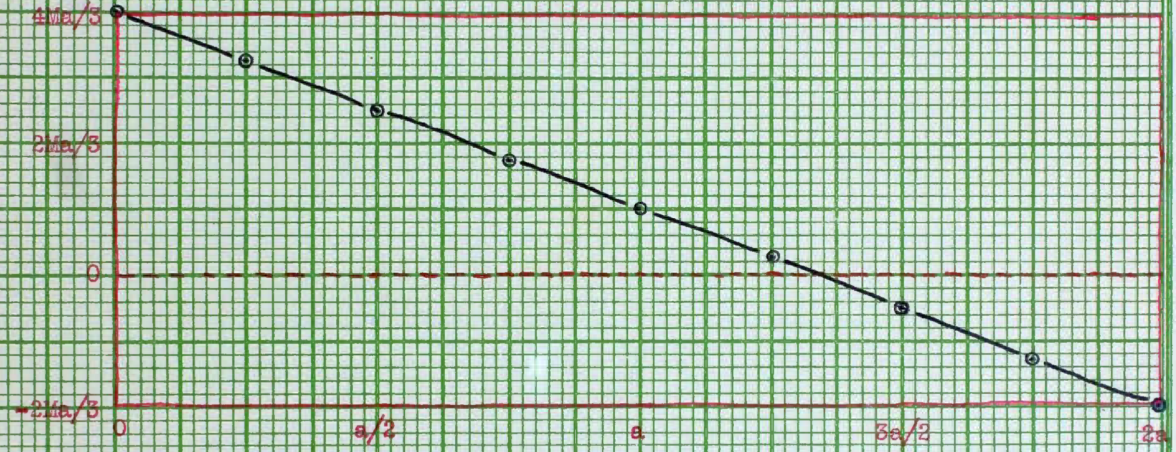


Fig. 9 The Heterogeneous Bar (σ varies as d)

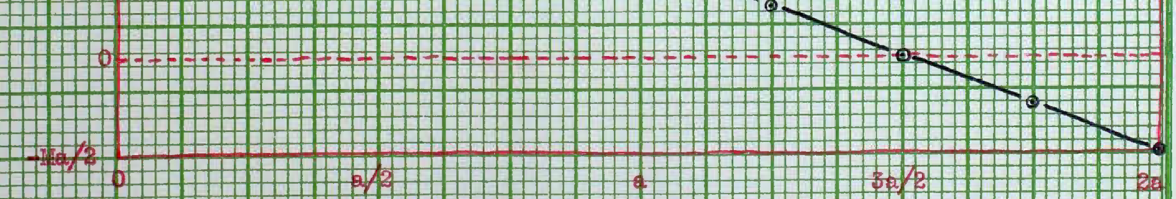


Fig. 10 The Heterogeneous Bar (ϕ varies as d^2)

VARIATION OF FIRST MOMENTS (CONT.)

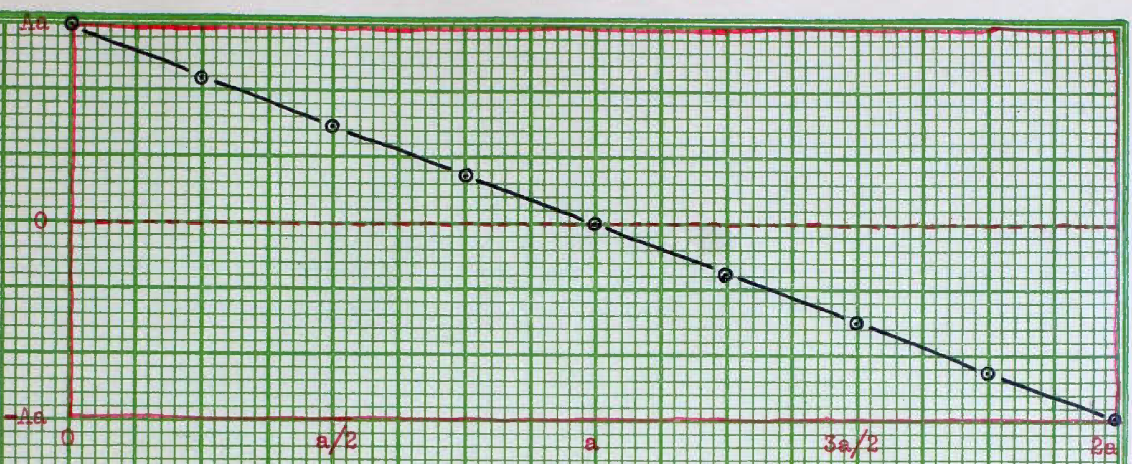


Fig. 11 The Ellipse

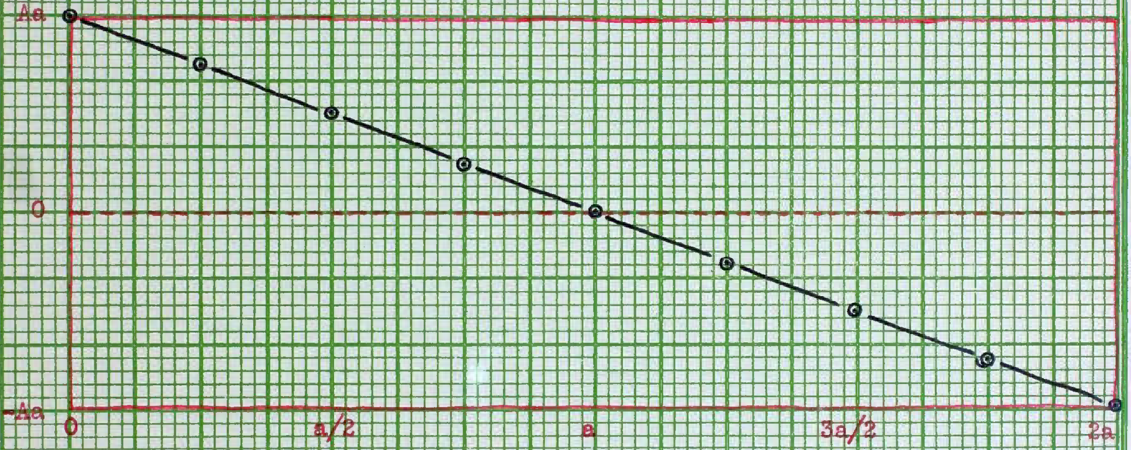


Fig. 12 The Circle

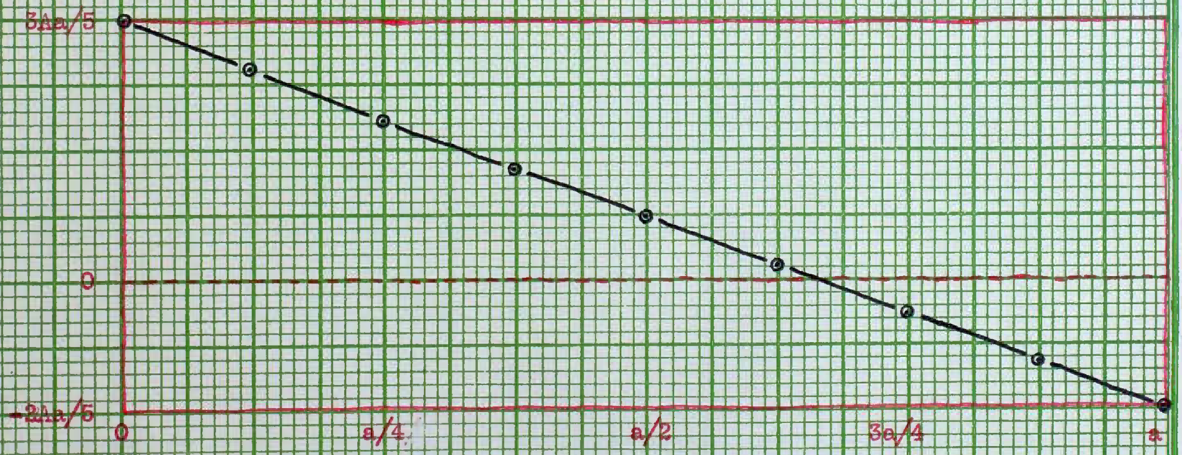


Fig. 13 The Parabola

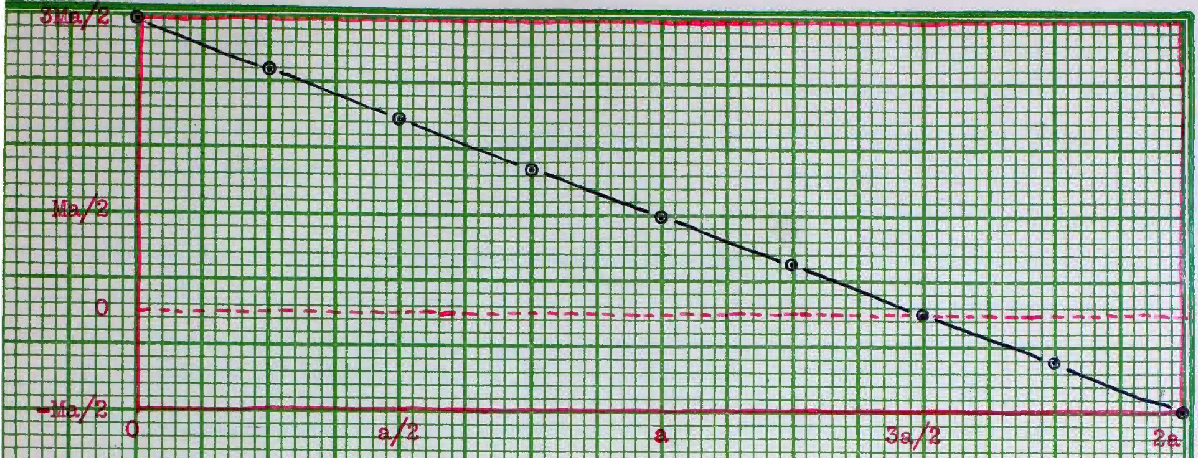


Fig. 14 The Cone

CHAPTER II

Variation of Second Moments

The moment of inertia, I_{ℓ}, of a body about any axis equals the moment of inertia, I_g, of the body about a parallel gravity axis plus the mass of the body multiplied by the square of the distance between these axes.[1]

The gravity axis passes through the center of area (or centroid). To calculate the center of mass of a semi-circular flat plate of radius a as in Fig. 15, we take the axes.

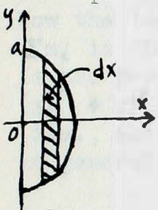


Fig. 15

From symmetry the center of mass lies on the $\underline{x}$ axis, i.e., as we have chosen our axes

$$\bar{y} = 0$$

$$\bar{x} = \int x \, dm / \int dm \qquad \text{II - 1}$$

Now dm = P dA, where d A is an area element and P is the superficial density (supposed in this case constant). Then $\bar{x}$ reduces to the center of area (or centroid as it is often called), viz.,

$$\bar{x} = \int x \, dA/A \qquad \text{II - 2}$$

where A is the area of the plate.[2]

Lindsay[3] also shows the development of the general gorm for finding the moment of inertia.

A special case of a general law called the theorem of parallel axes may be stated as follows: The moment of inertia of a rigid body about any axis is equal to the moment of inertia about a parallel axes through the center of mass plus the product of the mass of the body and the square of the perpendicular distance between the two axes. From Fig. 16 we may prove the theorem for the general case by taking the z axis as the axis

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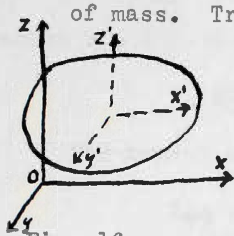
1. Miller and Lilly. Analytic Mechanics, p. 214.
2. Lindsay. Physical Mechanics, p. 152.
3. Lindsay. Physical Mechanics, p. 175-176.

of rotation and letting C with coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ be the center of mass. Treating the body as a set of mass particles, let us suppose that a particle m with coordinates x_i, y_i, z_i, in the original system has the coordinates x_i', y_i', z_i' when referred to a set of parallel axes through C. Now since $\bar{x}' = \bar{y}' = \bar{z}' = 0$, we must have



$$\Sigma m_i x_i' = \Sigma m_i y_i' = \Sigma m_i z_i' \quad , \quad \text{II - 3}$$

from the definition of center of mass. By definition the moment of inertia about the z axis is

Fig. 16

$$I = \Sigma m_i (x_i^2 + y_i^2) \qquad \text{II - 4}$$

$$x_i = x_i' - \bar{x}, \quad y_i = y_i' + \bar{y}, \quad \text{II - 5}$$

and hence

$$I = \Sigma m_i (x_i'^2 + y_i'^2) + (\bar{x}^2 + \bar{y}^2)\Sigma m_i + 2\bar{x}\Sigma m_i x_i' + 2\bar{y}\Sigma m_i y_i' \quad \text{II -6}$$

Now the last two terms of II - 6 vanish by virtue of II - 3. Then Σm_i is the total mass of the body, while $\bar{x}^2 + \bar{y}^2 = d^2$, where d is the perpendicular distance between the z and z' axes. $\Sigma m_i \cdot (x_i'^2 + y_i'^2)$ is the moment of inertia with respect to the z' axis (i.e., axis through center of mass parallel to the z axis). Hence in general

$$I = I_c - md^2. \qquad \text{II - 7}$$

Rietz[4] gives the following method of finding higher moments:

The nth moment, μn, is defined by $\mu n = \int_{-\infty}^{\infty} x^n y\, dx$, where y dx is an element of area, and x the distance of that element from the y - axis.

The center of mass of a uniform bar of length a, Fig. 1, may be found by

$$\bar{x} = \int x\, dM / \int dM = \int kx\, dx \Big/ \int k dx = \left[\frac{kx^2/2}{kx}\right]_0^a$$

$$= \frac{ka^2/2}{ka} = a/2$$

Where M = ka and dM = kdx

To find the moment of inertia of the bar with respect to a plane perpendicular to its axis and through the center of gravity we have

$$I_c = \int_0^{a/2} kx^2\, dx + \int_0^{a/2} kx^2\, dx = 2k\int_0^{a/2} x^2\, dx$$

$$= 2k\left[x^3/3\right]_0^{a/2} = ka^3/12 = 1/12\, Ma^2$$

4. Rietz, H. L. Handbook of Mathematical Statistics, p. 15.

Using the general formula, II - 7, to find the moment of inertia about $x = 0$

$$I_0 = 1/12\ Ma^2 + M\ (a/2)^2 = 1/3\ Ma^2$$

The moment of inertia about $x = a/8$ is

$$I_{a/8} = 1/12\ Ma^2 + M\ (3a/8)^2 = 43/192\ Ma^2$$

The moment of inertia about $x = a/4$ is

$$I_{a/4} = 1/12\ Ma^2 + M\ (a/4)^2 = 7/48\ Ma^2$$

The moment of inertia about $x = 3a/8$ is

$$I_{3a/8} = 1/12\ Ma^2 + M\ (a/8)^2 = 19/192\ Ma^2$$

Similarly the moments of inertia, Fig. 17, about planes on the opposite side of the center of mass are:

$$I_{5a/8} = 1/12\ Ma^2 + M\ (a/8)^2 = 19/192\ Ma^2$$

$$I_{3a/4} = 1/12\ Ma^2 + M\ (a/4)^2 = 7/48\ Ma^2$$

$$I_a = 1/12\ Ma^2 + M\ (a/2)^2 = 1/3\ Ma^2$$

In contrast to the homogeneous bar, the heterogeneous bar, Fig. 2, whose mass varies as the distance from one end, will be considered.

$$\sigma = kx = (2a)\,\sigma_{av} \qquad \sigma_{av} = ka$$

$$M = 2a\ (ka) = 2\ ka^2 \qquad dM = kx\ dx$$

$$M = \int_0^{2a} kx\ dx = 2\ ka^2$$

Since this is a heterogeneous bar, the general formula used for the homogeneous bar will not be applied; but each moment, Fig. 18, will be worked out separately.

The moment of inertia with respect to a plane through the origin is

$$I_y = \int_0^{2a} x^2\ (kx\ dx) = \left[kx^4/4\right]_0^{2a} = 4\ ka^4 = 2\ Ma^2$$

About a/4

$$I_y = \int_0^{2a} (x - a/4)^2\ kx\ dx = \left[kx^4/4 - kax^3/6 + ka^2x^2/32\right]_0^{2a}$$

$$= 61/24\ ka^4 = 61/48\ Ma^2$$

About a/2

$$I_y = \int_0^{2a} (x - a/2)^2 kx\ dx = \left[kx^4/4 - kax^3/3 + ka^2x^2/8\right]_0^{2a}$$

$$= 11/6\ ka^4 = 11/12\ Ma^2$$

About 3a/4

$$I_y = \int_0^{2a} (x - 3a/4)^2\ kx\ dx = \left[kx^4/4 - kax^3/2 + 9ka^2x^2/32\right]_0^{2a}$$

$$= 9/8\ ka^4 = 9/16\ Ma^2$$

About a

$$I_y = \int_0^{2a} (x - a)^2\ kx\ dx = \left[kx^4/4 - 2kax^3/3 + ka^2x^2/2\right]_0^{2a}$$

$$= 2/3\ ka^4 = 1/3\ Ma^2$$

About 5a/4

$$I_y = \int_0^{2a} (x - 5a/4)^2\ kx\ dx = \left[kx^4/4 - 5kax^3/6 + 25ka^2x^2/32\right]_0^{2a}$$

$$= (11/24)\ ka^4 = (11/48)\ Ma^2$$

About 3a/2

$$I_y = \int_0^{2a} (x - 3a/2)^2\ kx\ dx = \left[kx^4/4 - kax^3 + 9ka^2x^2/8\right]_0^{2a}$$

$$= 1/2\ ka^4 = 1/4\ Ma^2$$

About 7a/4

$$I_y = \int_0^{2a} (x - 7a/4)^2\ kx\ dx = \left[kx^4/4 - 7kax^3/6 + 49ka^2x^2/32\right]_0^{2a}$$

$$= 19/24\ ka^4 = 19/48\ Ma^2$$

About 2a

$$I_y = \int_0^{2a} (x - 2a)^2\ kx\ dx = \left[kx^4/4 - 4kax^3/3 + 2ka^2x^2\right]_0^{2a}$$

$$= 4/3\ ka^4 = 2/3\ Ma^2$$

Next the moments of inertia, Fig. 19, of the heterogeneous bar,

Fig. 3, whose mass varies as the square of the distance from one end will be considered.

Let 2a = length and ϕ = density

$$\phi = kx \qquad\qquad dM = kx^2\, dx$$

$$M = \int_0^{2a} kx^2 dx = \left[kx^3/3\right]_0^{2a} = 8/3\ ka^3$$

The moment of inertia with respect to a plane through the origin is

$$I_y = \int_0^{2a} x^2(kx^2 dx) = \left[kx^5/5\right]_0^{2a} = 32/5\ ka^5 = 12/5\ Ma^2$$

About a/4

$$I_y = \int_0^{2a} (x - a/4)^2\ kx^2\ dx = \left[kx^5/5 - kax^4/8 + ka^2x^3/48\right]_0^{2a}$$

$$= 137/30\ ka^5 = 137/80\ Ma^2$$

About a/2

$$I_y = \int_0^{2a} (x - a/2)^2\ kx^2\ dx = \left[kx^5/5 - kax^4/4 + ka^2x^3/12\right]_0^{2a}$$

$$= 46/15\ ka^5 = 23/20\ Ma^2$$

About 3a/4

$$I_y = \int_0^{2a} (x - 3a/4)^2\ kx^2\ dx = \left[kx^5/5 - 3kax^4/8 + 3ka^2x^3/16\right]_0^{2a}$$

$$= 19/10\ ka^5 = 57/80\ Ma^2$$

About a

$$I_y = \int_0^{2a} (x - a)^2\ kx^2\ dx = \left[kx^5/5 - kax^4/2 + ka^2x^3/3\right]_0^{2a}$$

$$= 16/15\ ka^5 = 2/5\ Ma^2$$

About 5a/4

$$I_y = \int_0^{2a} (x - 5a/4)^2\ kx^2\ dx = \left[kx^5/5 - 5kax^4/8 + 25ka^2x^3/48\right]_0^{2a}$$

$$= 17/30\ ka^5 = 17/80\ Ma^2$$

About 3a/2

$$I_y = \int_0^{2a} (x - 3a/2)^2\ kx^2\ dx = \left[kx^5/5 - 3kax^4/4 + 3ka^2x^3/4\right]_0^{2a}$$

$$= 2/5\ ka^5 = 6/40\ Ma^2$$

About 7a/4

$$I_y = \int_0^{2a} (x - 7a/4)^2 \; kx^2 \, dx = \left[kx^5/5 - 7kax^4/8 + 49ka^2x^3/48 \right]_0^{2a}$$

$$= 17/30 \; ka^5 = 17/80 \; Ma^2$$

About 2a

$$I_y = \int_0^{2a} (x - 2a)^2 \; kx^2 \, dx = \left[kx^5/5 - kax^4 + 4ka^2x^3/3 \right]_0^{2a}$$

$$= 16/15 \; ka^5 = 2/5 \; Ma^2$$

To find the moments of inertia, Fig. 20, of an ellipse, Fig. 4, the general form is used.

$$\text{If } y = \pm \, b/a \sqrt{2ax - x^2}$$

$$A = 2 \int_0^{2a} b/a \sqrt{2ax - x^2} \, dx$$

$$= ab \, (\pi/2 - \pi/2) = ab\pi$$

The moments of inertia about <u>a</u>, a line through the center of gravity, is found by

$$I_C = 2 \int_0^{2a} (x - a)^2 \; b/a \sqrt{2ax - x^2} \, dx$$

Which when integrated becomes

$$I_C = \Big[(1 - x/a) \, b/2 \sqrt{(2ax - x^2)^3} + ab \, (x - a)/4 \sqrt{2ax - x^2} + a^3b/4 \sin^{-1} (x - a)/a \Big]_0^{2a}$$

$$= 0 + 0 + a^3b/4 \, (\pi) = 1/4 \; Aa^2$$

The moment of inertia about $x = 0$ is

$$I_0 = 1/4 \; Aa^2 + Aa^2 = 5/4 \; Aa^2$$

The moment of inertia about $x = a/4$ is

$$I_{a/4} = 1/4 \; Aa^2 + A \, (9a^2/16) = 13/16 \; Aa^2$$

The moment of inertia about $x = a/2$ is

$$I_{a/2} = 1/4 \; Aa^2 + A \, (a^2/4) = 1/2 \; Aa^2$$

The moment of inertia about 3a/4 is

$$I_{3a/4} = 1/4 \; Aa^2 + A \, (a^2/16) = 5/16 \; Aa^2$$

Similarly the moments of inertia about planes on the opposite side of the center of mass are:

$$I_{5a/4} = 1/4\ Aa^2 + A\ (a^2/16) = 5/16\ Aa^2$$

$$I_{3a/2} = 1/4\ Aa^2 + A\ (a^2/4) = 1/2\ Aa^2$$

$$I_{7a/4} = 1/4\ Aa^2 + A\ (9a^2/16) = 13/16\ Aa^2$$

$$I_{2a} = 1/4\ Aa^2 + A\ (a^2) = 5/4\ Aa^2$$

The moments of inertia, Fig. 21, of a circle, Fig. 5, which is a special form of an ellipse gives the same results as those found for the ellipse when the limits of each are 0 and 2a.

$$\text{If } y = \pm\sqrt{2ax - x^2}$$

$$A = 2\int_0^{2a}\sqrt{2ax - x^2}\,dx = \pi a^2$$

The moment of inertia about a line through a, the center of gravity, is

$$I_C = 2\int_0^{2a}(x - a)^2\sqrt{2ax - x^2}\,dx = 1/4\ Aa^2$$

The other moments also are the same as the ellipse

$$I_0 = 1/4\ Aa^2 + Aa^2 = 5/4\ Aa^2$$

$$I_{a/4} = 1/4\ Aa^2 + A\ (9/16\ a^2) = 13/16\ Aa^2$$

$$I_{a/2} = 1/4\ Aa^2 + A\ (a^2/4) = 1/2\ Aa^2$$

$$I_{3a/4} = 1/4\ Aa^2 + A\ (a^2/16) = 5/16\ Aa^2$$

$$I_{3a/2} = 1/4\ Aa^2 + A\ (a^2/4) = 1/2\ Aa^2$$

$$I_{7a/4} = 1/4\ Aa^2 + A\ (9/16a^2) = 13/16\ Aa^2$$

$$I_{2a} = 1/4\ Aa^2 + A\ (a^2) = 5\ Aa^2/4$$

The moments of inertia, Fig. 22, of a parabola, Fig. 6, may be found by use of the general formula if the center of gravity of a definite part of the body is found.

The center of gravity of that part of the parabola between 0 and a is on a line about which the first moment of this much of the parabola equals 0.

The equation of the parabola is $y^2 = 4ax$ and the line through the center of gravity was found to be 3/5 a; therefore the moment of inertia about a line through the center of gravity is

$$I_c = 2\int_0^{2a} (x - 3/5\ a)^2\ 2\sqrt{ax}\ dx = 32/175\ a^4 = 12/175\ Aa^2$$

About 0 the moment of inertia is

$$I_o = 12/175\ Aa^2 + A\ (9/25\ a^2) = 75/175\ Aa^2$$

The other moments about perpendicular lines along the axis are

$$I_{a/10} = 12/175\ Aa^2 + A\ (1/4\ a^2) = 223/700\ Aa^2$$

$$I_{a/5} = 12/175\ Aa^2 + A\ (4/25\ a^2) = 40/175\ Aa^2$$

$$I_{3a/10} = 12/175\ Aa^2 + A\ (9/100\ a^2) = 101/700\ Aa^2$$

$$I_{2a/5} = 12/175\ Aa^2 + A\ (1/25\ a^2) = 19/175\ Aa^2$$

$$I_{a/2} = 12/175\ Aa^2 + A\ (1/100\ a^2) = 55/700\ Aa^2$$

$$I_{7a/10} = 12/175\ Aa^2 + A\ (1/100\ a^2) = 55/700\ Aa^2$$

$$I_{4a/5} = 12/175\ Aa^2 + A\ (1/25\ a^2) = 19/175\ Aa^2$$

$$I_{9a/10} = 12/175\ Aa^2 + A\ (9/100\ a^2) = 101/700\ Aa^2$$

$$I_a = 12/175\ Aa^2 + A\ (4/25\ a^2) = 40/175\ Aa^2$$

The moments of inertia, Fig. 23, of the mass of a cone, Fig. 7, about planes along the axis may also be found by the general formula. Since the first moment about 3a/2 was found to be 0, the center of gravity lies on a line through 3a/2.

It was found that $y^2 = h^2x^2/4a^2$ and that

$M = \int_0^{2a} \pi y^2\ dx = 2/3\ \pi a\ h^2$ in Chapter I.

The moment of inertia about a plane through 3a/2 is

$$I_c = \int_0^{2a} (x - 3/2\ a)^2\ \pi y^2\, dx = \left[\pi h^2 x^5/20a^2 - 3\pi h^2 x^4/16a + 3\pi h^2 x^3/16\right]_0^{2a}$$

$$= 1/10\ \pi h^2 a^3 = 3\ Ma^2/20$$

By the general formula the following moments may be found:

$$I_o = 3/20\ Ma^2 + M\ (9/4\ a^2) = 12/5\ Ma^2$$

$$I_{a/4} = 3/20\ Ma^2 + M\ (25/16\ a^2) = 137/80\ Ma^2$$

$$I_{a/2} = 3/20\ Ma^2 + M\ (a^2) = 23/20\ Ma^2$$

$$I_{3a/4} = 3/20\ Ma^2 + M\ (9/16\ a^2) = 57/80\ Ma^2$$

$$I_a = 3/20\ Ma^2 + M\ (a^2/4) = 2/5\ Ma^2$$

$$I_{5a/4} = 3/20\ Ma^2 + M\ (a^2/16) = 17/80\ Ma^2$$

$$I_{7a/4} = 3/20\ Ma^2 + M\ (a^2/16) = 17/80\ Ma^2$$

$$I_{2a} = 3/20\ Ma^2 + M\ (a^2/4) = 2/5\ Ma^2$$

The following graphs illustrate the results obtained for the second moments:

VARIATION OF SECOND MOMENTS

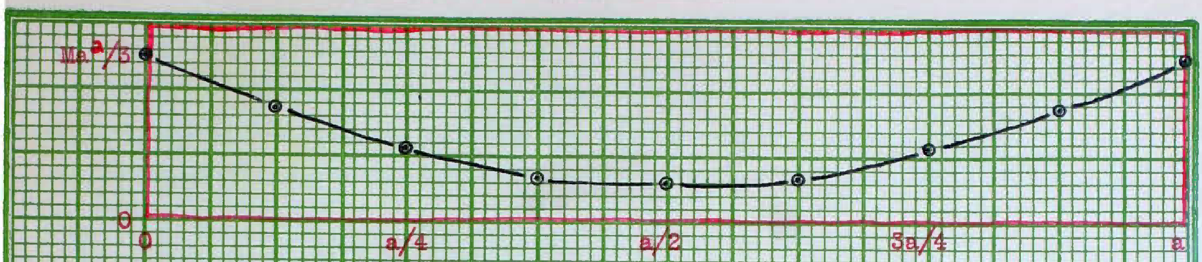


Fig. 17 The Homogeneous Bar

Fig. 18 The Heterogeneous Bar (σ varies as d)

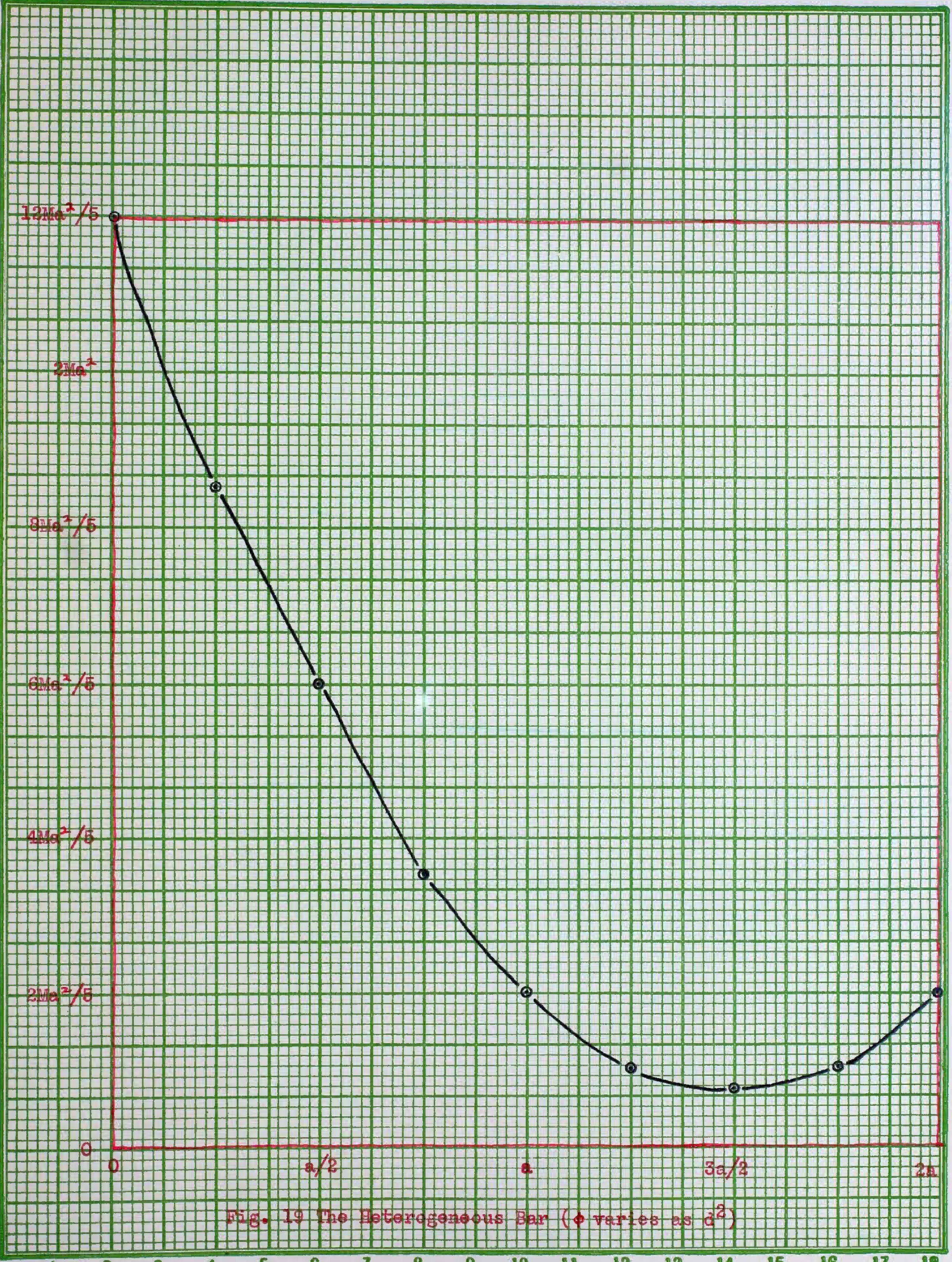


Fig. 19 The Heterogeneous Bar (ϕ varies as d^2)

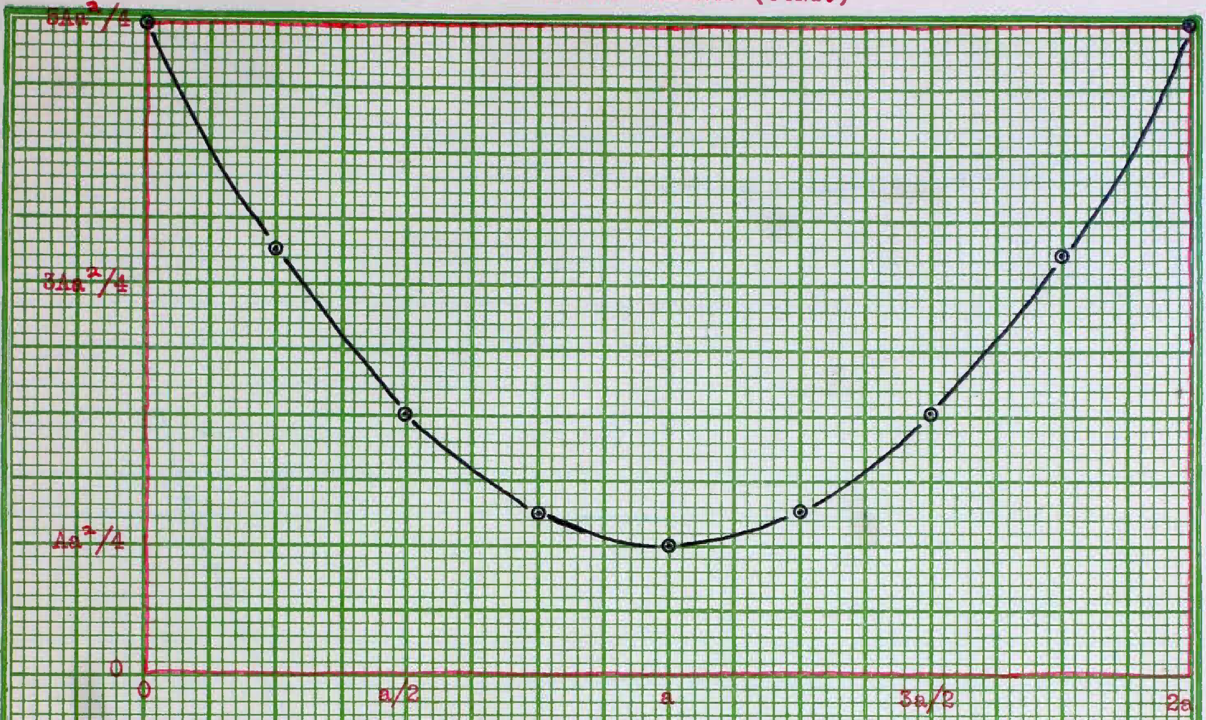


Fig. 20 The General Ellipse

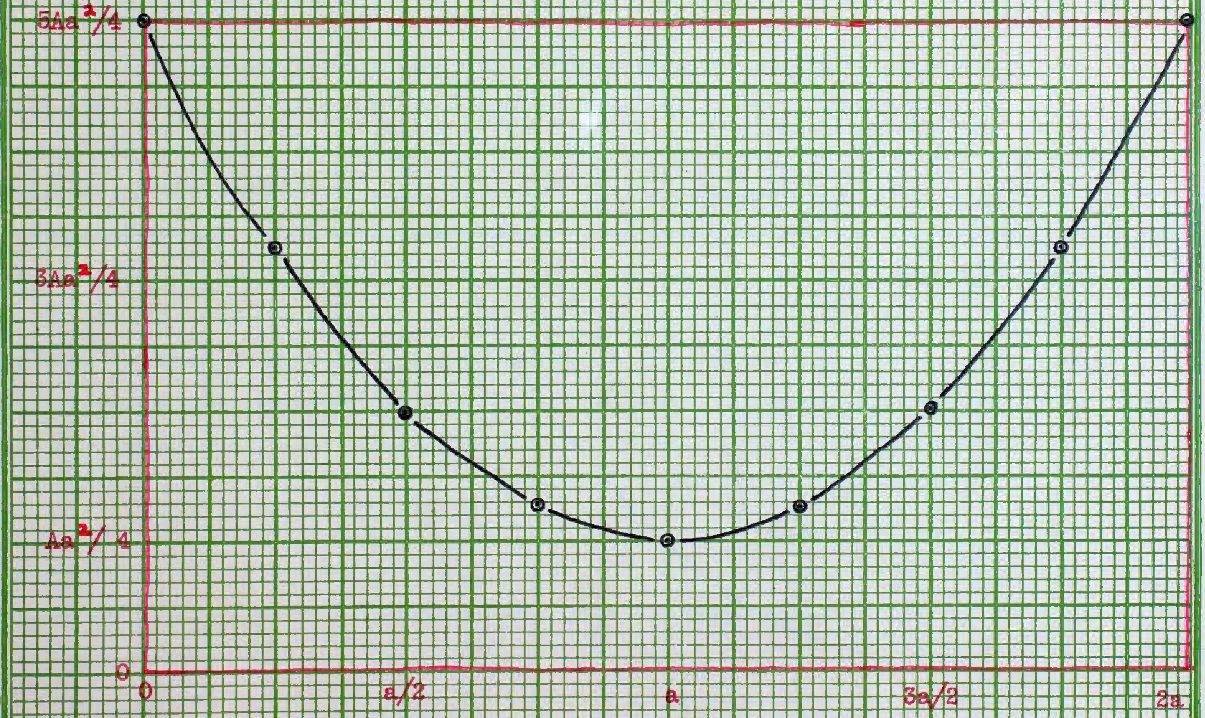


Fig. 21 The Circle

VARIATION OF SECOND MOMENTS (CONT.)

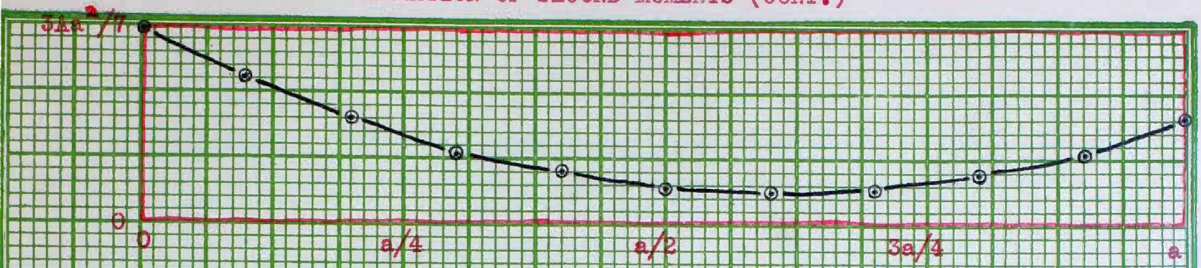


Fig. 22 The Parabola

Fig. 23 The Cone

CHAPTER III

Variation of Third Moments

In statistics the third moment about the mean of a distribution curve is a measure of the skewness of the curve; however, since the general formula for finding the third moment is

$$I_y = \int x^3 y \, dx,$$

The moment may be found about planes through other points besides the mean.

The third moments of the uniform bar, Fig. 1 and 24, may be found in the following manner:

About the origin $\quad I_y = \int_0^a x^3 k \, dx = ka^4/4 = Ma^3/4$

About a/8 $\quad I_y = \int_{a/8}^a k(x - a/8)^3 \, dx + \int_0^{a/8} k\,(x - a/8)^3 \, dx$

$$\text{or } I_y = \int_0^a k(x - a/8)^3 \, dx = k \int (x^3 - 3/8\, ax^2 + 3/64\, a^2 x - a^3/512) \, dx$$

$$= \Big[k(x^4/4 - 3ax^3/24 - 3a^2x^2/128 - a^3x/512\Big]_0^a =$$

$$= \big[k\,(a^4/4 - 3a^4/24 + 3a^4/128 - a^4/512) = 450ka^4/3072$$

$$= 75Ma^3/512$$

A similar method was used to find the moments about planes through other points along the axis.

About a/4 $\qquad I_y = 5Ma^3/64$

About 3a/8 $\qquad I_y = 17Ma^3/512$

About $a/2$ $\qquad I_y = 0$

About $5a/8$ $\qquad I_y = -17Ma^3/512$

About $3a/4$ $\qquad I_y = -5Ma^3/64$

About $7a/8$ $\qquad I_y = -75Ma^3/512$

About a $\qquad I_y = -Ma^3/4$

Third moments of a heterogeneous bar whose mass varies as the distance from one end, Fig. 2 and 25, are worked out by the same method as that used for the uniform bar.

About the origin $\qquad I_y = \int_0^{2a} x^3\, kx\, dx$

$$= \left[kx^5/5\right]_0^{2a} = 32ka^5/5 = 16Ma^3/5$$

About $a/4$ $\qquad I_y = \int_0^{2a} (x - a/4)^3\, kx\, dx = 619Ma^3/320$

About $a/2$ $\qquad I_y = 43Ma^3/40$

About $3a/4$ $\qquad I_y = 169\, Ma^3/320$

About a $\qquad I_y = Ma^3/5$

About $5a/4$ $\qquad I_y = -Ma^3/320$

About $3a/2$ $\qquad I_y = -7Ma^3/40$

About $7a/4$ $\qquad I_y = -131Ma^3/320$

About 2a $\qquad I_y = -4Ma^3/5$

Different results will be obtained when the third moments for the heterogeneous bar whose mass varies as the square of the distance from one end, Fig. 3 and 25, are computed.

About the origin $\qquad I_y = \int_0^{2a} x^3\, kx^2\, dx$

$$= \left[kx^6/6\right]_0^{2a} = 32ka^6/3 = 4\, Ma^3$$

About $a/4$ $\qquad I_y = \int_0^{2a} (x - a/4)^3\, kx^2\, dx = 789Ma^3/320$

About $a/2$ $\quad I_y = 7Ma^3/5$

About $3a/4$ $\quad I_y = 227Ma^3/320$

About a $\quad I_y = 3Ma^3/10$

About $5a/4$ $\quad I_y = 5Ma^3/64$

About $3a/2$ $\quad I_y = -Ma^3/20$

About $7a/4$ $\quad I_y = -57Ma^3/320$

About $2a$ $\quad I_y = -2Ma^3/5$

Since the general form of the ellipse also includes the circle, the third moments were not computed for the circle but only for the ellipse, Fig. 4 and 27.

The third moment about a line through the origin is

$$I_y = 2\int_0^{2a} x^3 (b/a) \sqrt{2ax - x^2}\, dx$$

$$= 2b/a \left[- x^2 \sqrt{(2ax - x^2)^3}\ /5 \right]_0^{2a} + 7a/5 \int_0^{2a} x^2 \sqrt{2ax - x^2}\, dx$$

which, when completely integrated and with the limits substituted in the result becomes $7\pi a^4 b/4 = 7Aa^3/4$

About $a/8$ $\quad I_y = 2\int_0^{2a} (x - a/8)^3 \ (b/a) \sqrt{2ax - x^2}\, dx$

$\quad = 679Aa^3/512$

About $a/4$ $\quad I_y = 63Aa^3/64$

About $3a/8$ $\quad I_y = 365Aa^3/512$

About $a/2$ $\quad I_y = 7Aa^3/12$

About $5a/8$ $\quad I_y = 171Aa^3/512$

About $3a/4$ $\quad I_y = 13Aa^3/64$

About $7a/8$ $\quad I_y = 49Aa^3/512$

About a $\quad I_y = 0$

About $9a/8$ $\quad I_y = -49Aa^3/512$

About 5a/4 $\quad I_y = -13Aa^3/64$

About 11a/8 $\quad I_y = -171Aa^3/512$

About 3a/8 $\quad I_y = -7Aa^3/12$

About 13a/8 $\quad I_y = -365Aa^3/512$

About 7a/4 $\quad I_y = -63Aa^3/64$

About 15a/8 $\quad I_y = -679Aa^3/512$

About 2a $\quad I_y = -7Aa^3/4$

The parabola is also a suitable figure to use in studying the third moments, Fig. 6 and 28.

The third moment about a line through the origin is determined by

$$I_y = 2\int_0^a x^3\, 2\sqrt{ax}\, dx = \left[8\sqrt{a}/9\,(x)^{9/2}\right]_0^a$$

$$= 8a^5/9 = Aa^3/3$$

About a/10 $\quad I_y = 2\int_0^a (x - a/10)^3\; 2\sqrt{ax}\, dx = 4657Aa^3/21000$

About a/5 $\quad I_y = 368Aa^3/2625$

About 3a/10 $\quad I_y = 1735Aa^3/21000$

About 2a/5 $\quad I_y = 113Aa^3/2625$

About a/2 $\quad I_y = 13Aa^3/840$

About 3a/5 $\quad I_y = -16Aa^3/2625$

About 7a/10 $\quad I_y = -83Aa^3/3000$

About 4a/5 $\quad I_y = -145Aa^3/2625$

About 9a/10 $\quad I_y = -1991Aa^3/21000$

About a $\quad I_y = -16Aa^3/105$

Lastly, the third moments of the cone, Fig. 7 and 29, will be considered.

About a plane through the origin the third moment of the cone is

$$I_y = \int_0^{2a} x^3 \pi (h^2 x^2/4a^2)\,dx = \left[\pi h^2 x^6/24a^2\right]_0^{2a}$$

$$= 8\pi h^2 a^4/3 = 4Ma^3$$

About $a/4$	$I_y = \int_0^{2a} (x - a/4)^3 \ \pi\, y^2 dx = 789Ma^3/320$
About $a/2$	$I_y = 7Ma^3/5$
About $3a/4$	$I_y = 227Ma^3/320$
About a	$I_y = 3Ma^3/10$
About $5a/4$	$I_y = 5Ma^3/64$
About $3a/2$	$I_y = -Ma^3/20$
About $7a/4$	$I_y = -57Ma^3/320$
About $2a$	$I_y = -2Ma^3/5$

The following graphs illustrate the results that have been obtained for third moments:

VARIATION OF THIRD MOMENTS

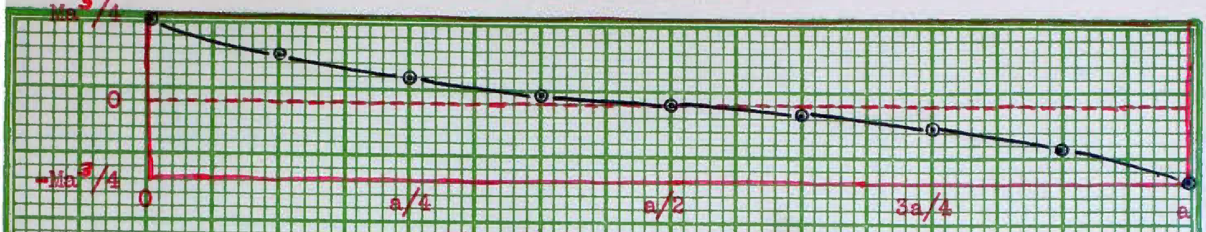


Fig. 24 The Homogeneous Bar

Fig. 25 The Heterogeneous Bar (σ varies as d)

VARIATION OF THIRD MOMENTS (CONT.)

Fig. 26 The Heterogeneous Bar (ρ varies as d^2)

VARIATION OF THIRD MOMENTS (CONT.)

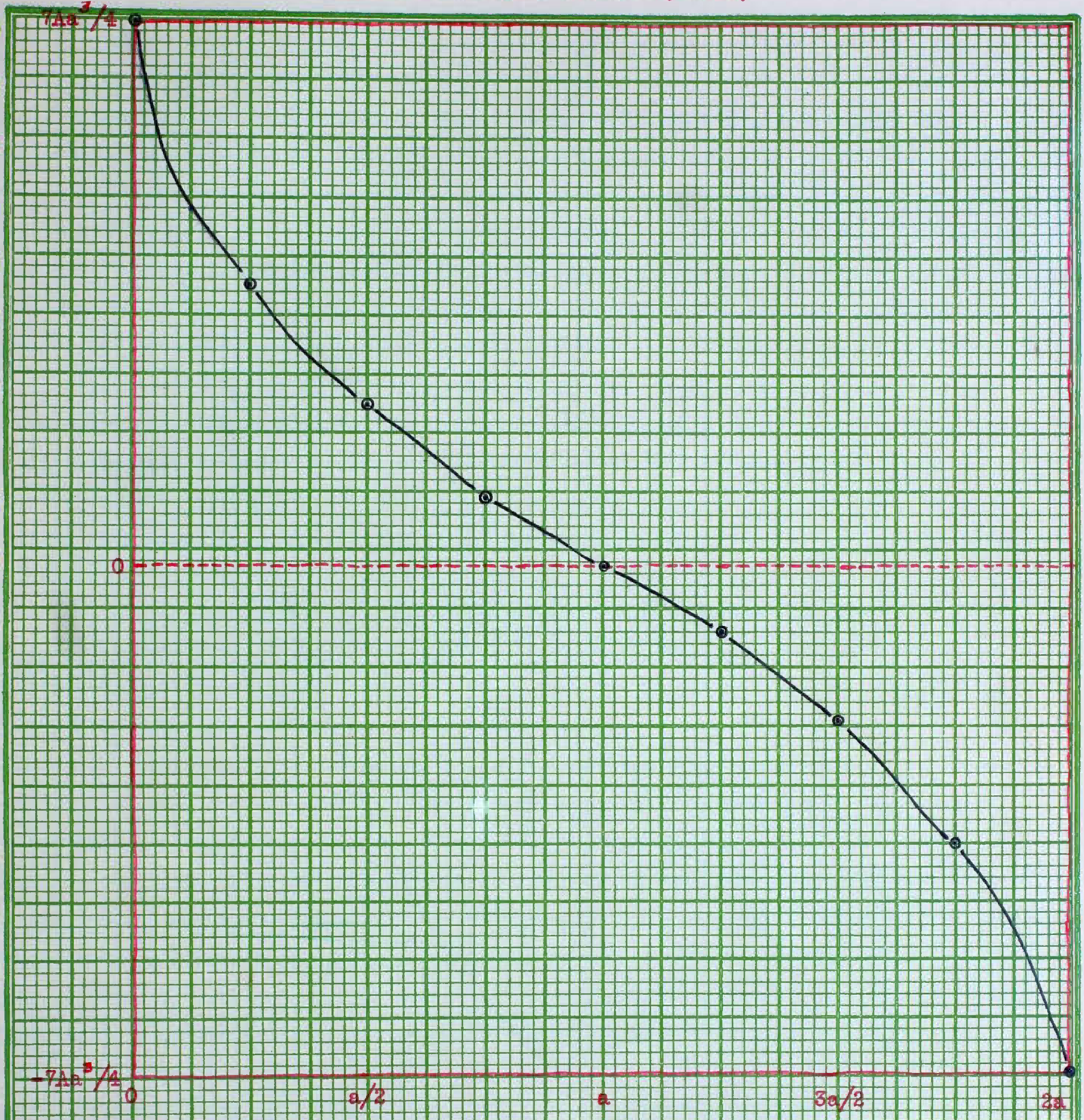


Fig. 27 The Ellipse

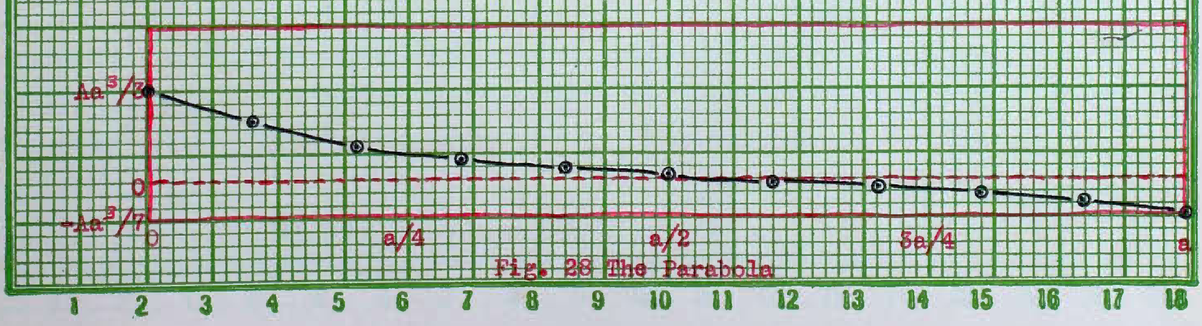


Fig. 28 The Parabola

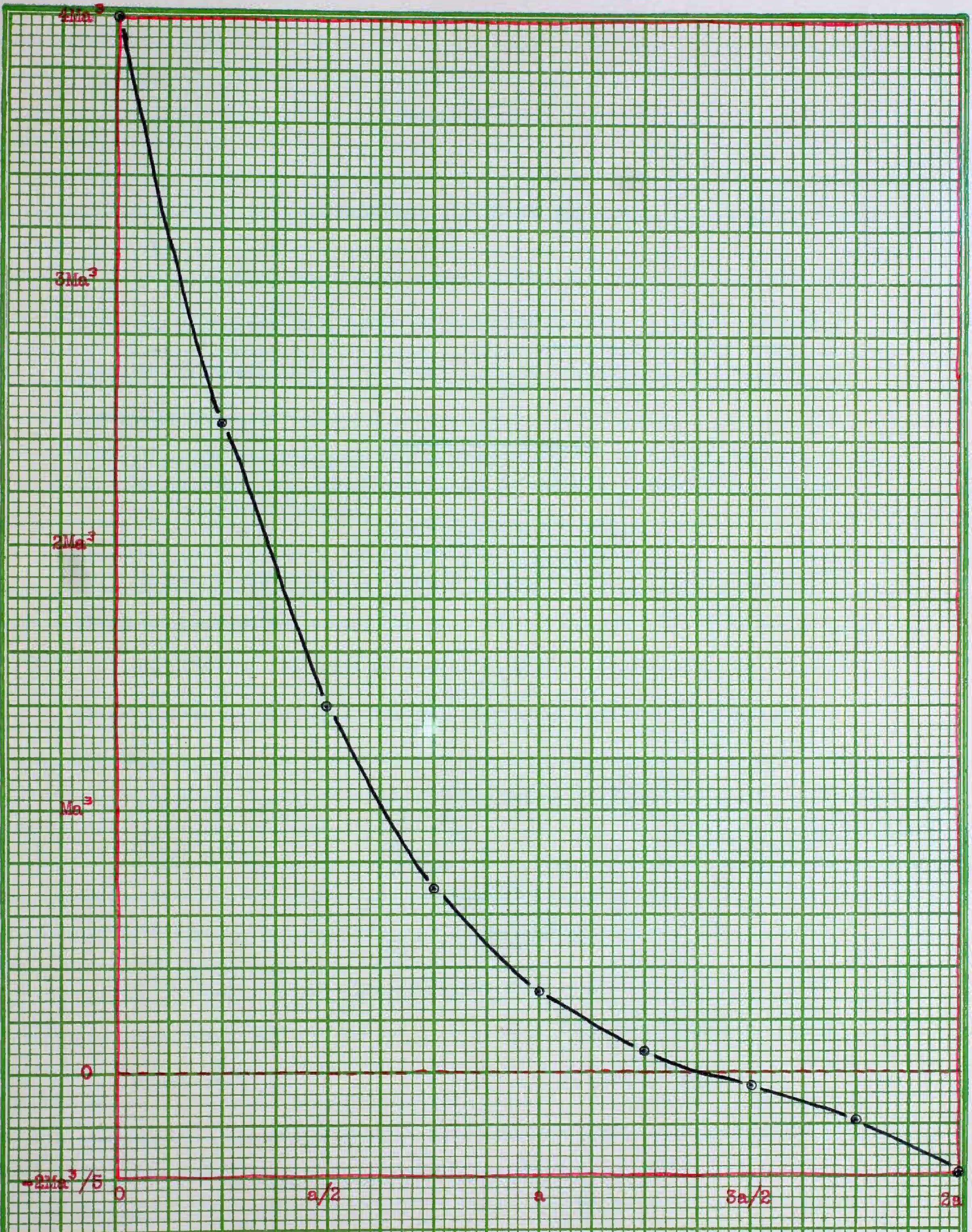


Fig. 29 The Cone

CHAPTER IV

Variation of Fourth Moments

In statistics the fourth moment about the mean of a distribution curve is a measure of the kurtosis of the curve. Harper[1], however, gives a different method of obtaining the kurtosis of a frequency distribution which is

> A normal distribution has a kurtosis of 3, which may be calculated by dividing the first moment to the fourth power by the square of the second moment
>
> $$(\text{Kurtosis})\ B_2 = M^4/M_2^2$$

By use of the general formula $I_y = \int x^4 y\, dx$, the fourth moments about various points along the axis of a body may be found.

Considering the uniform bar, Fig. 1, the fourth moments, Fig. 30, about planes at various intervals along the axis may be computed as follows:

About the origin $\quad I_y = \int_0^a kx^4\, dx = \left[kx^5/5\right]_0^a = Ma\ /5$

About a/8 $\quad I_y = \int_0^a k(x - a/8)^4\, dx = \int_0^a k(x^4 - ax^3/2 + 3a^2x^2/32 - a^3x/128 + a^4/4096)\, dx$

or $I_y = 2101Ma^4/20{,}480$

By integration of similar forms other moments are found to be

About a/4 $\quad I_y = 61Ma^4/1280$

About 3a/8 $\quad I_y = 421Ma^4/20{,}480$

---

1. Harper. Elements of Practical Statistics, p. 151.

About $a/2$ $\qquad I_y = Ma^4/80$

About $5a/8$ $\qquad I_y = 421Ma^4/20{,}480$

About $3a/4$ $\qquad I_y = 61Ma^4/1280$

About $7a/8$ $\qquad I_y = 2101Ma^4/20{,}480$

About a $\qquad I_y = Ma^4/5$

The fourth moments of a heterogeneous bar whose mass varies as the distance from one end, Fig. 2, differ from the results of the homogeneous bar since the results for the heterogeneous bar when plotted, Fig. 31, do not determine a regular curve.

The fourth moment about a plane through the origin may be determined by

$$I_y = \int_0^{2a} x^4\, kx\, dx = \left[kx^6/6\right]_0^{2a} = 32ka^6/3 = 16Ma^4/3$$

By integration of similar forms the moment may be found at any interval along the axis.

About $a/4$ $\qquad I_y = 3589Ma^4/1280$

About $a/2$ $\qquad I_y = 329Ma^4/240$

About $3a/4$ $\qquad I_y = 2111Ma^4/3840$

About a $\qquad I_y = Ma^4/5$

About $5a/4$ $\qquad I_y = 83Ma^4/768$

About $3a/2$ $\qquad I_y = 47Ma^4/240$

About $7a/4$ $\qquad I_y = 1839Ma^4/3840$

About $2a$ $\qquad I_y = 16Ma^4/15$

Interesting results may also be obtained by computing the fourth moments for a heterogeneous bar whose mass varies as the square of the distance from one end, Fig. 3 and 32.

The fourth moment about a plane through the origin may be determined by

$$I_y = \int_0^{2a} x^4 kx^2\,dx = \left[kx^7/7\right]_0^{2a} = 128ka^7/7 = 48Ma^4/7$$

By integration the moments were found about planes at various intervals along the axis.

About a/4	$I_y = 32859Ma^4/8960$
About a/2	$I_y = 991Ma^4/560$
About 3a/4	$I_y = 6651Ma^4/8960$
About a	$I_y = 9Ma^4/35$
About 5a/4	$I_y = 143Ma^4/1792$
About 3a/2	$I_y = 39Ma^4/560$
About 7a/4	$I_y = 1611Ma^4/8960$
About 2a	$I_y = 16Ma^4/35$

The fourth moments for an ellipse, Fig. 4 and 33, give results similar to those for the uniform bar since both are homogeneous bodies.

About a line through the origin the fourth moment for the ellipse is

$$I_y = 2\int_0^{2a} x^4 (b/a)\sqrt{2ax - x^2}\,dx$$

which, when expanded according to the general form in Pierce's Table of Integrals[1] and the limits substituted in, becomes

$$21\pi a^6/8 = 21Aa^4/8$$

By a similar process the following fourth moments for the ellipse may be found:

---

1. Pierce, B. O. A Short Table of Integrals. Third revised edition.

About a/4	$I_y = 43Aa^4/32$
About a/2	$I_y = 9Aa^4/16$
About 3a/4	$I_y = 57Aa^4/256$
About a	$I_y = Aa^4/8$
About 5a/4	$I_y = 57Aa^4/256$
About 3a/2	$I_y = 9Aa^4/16$
About 7a/4	$I_y = 43Aa^4/32$
About 2a	$I_y = 21Aa^4/8$

As another example of a heterogeneous body the fourth moment of the area of a definite portion of a parabola, Fig. 6 and 34, may be determined.

The fourth moment about a line through the origin is

$$I_y = 2\int_0^{2a} x^4 \, 2\sqrt{ax}\, dx = \left[8\sqrt{a}\, x^{11/2}/11\right]_0^a = 8a^6/11 = 3Aa^4/11$$

Similarly the following moments were obtained:

About a/10	$I_y = 71{,}049Aa^4/462{,}000$
About a/5	$I_y = 13{,}976Aa^4/144{,}375$
About 3a/10	$I_y = 111{,}223Aa^4/2{,}310{,}000$
About 2a/5	$I_y = 3295Aa^4/144{,}375$
About a/2	$I_y = 211Aa^4/18{,}480$
About 3a/5	$I_y = 33{,}108Aa^4/2{,}665{,}000$
About 7a/10	$I_y = 5377Aa^4/330{,}000$
About 4a/5	$I_y = 4833Aa^4/144{,}375$
About 9a/10	$I_y = 47{,}805Aa^4/770{,}000$
About a	$I_y = 128Aa^4/1155$

In the conclusion, the fourth moments of a cone, Fig. 7 and 35, have been computed.

The fourth moment of the mass of a cone about a plane through the origin is

$$I_y = \int_0^{2a} x^4 \pi \, (h^2 x^2/4a^2) \, dx = 32 \pi h^2 a^5/7 = 48Ma^4/7$$

Similarly about $a/4$	$I_y = 25{,}691Ma^4/8960$
About $a/2$	$I_y = 991Ma^4/560$
About $3a/4$	$I_y = 6651Ma^4/8960$
About a	$I_y = 9Ma^4/35$
About $5a/4$	$I_y = 143Ma^4/1792$
About $3a/2$	$I_y = 39Ma^4/560$
About $7a/4$	$I_y = 1611Ma^4/8960$
About $2a$	$I_y = 16Ma^4/35$

The following graphs illustrate the results that have been obtained for fourth moments:

VARIATION OF FOURTH MOMENTS

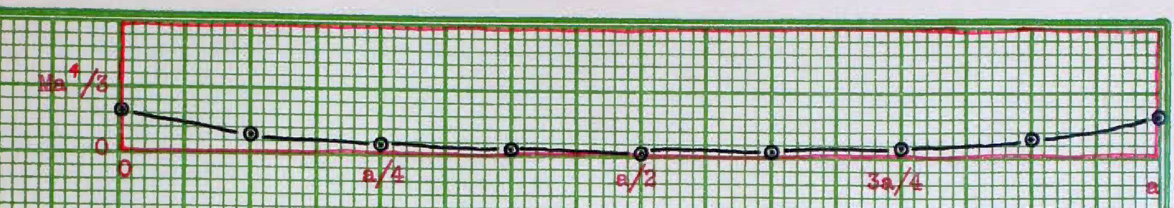


Fig. 30 The Homogeneous Bar

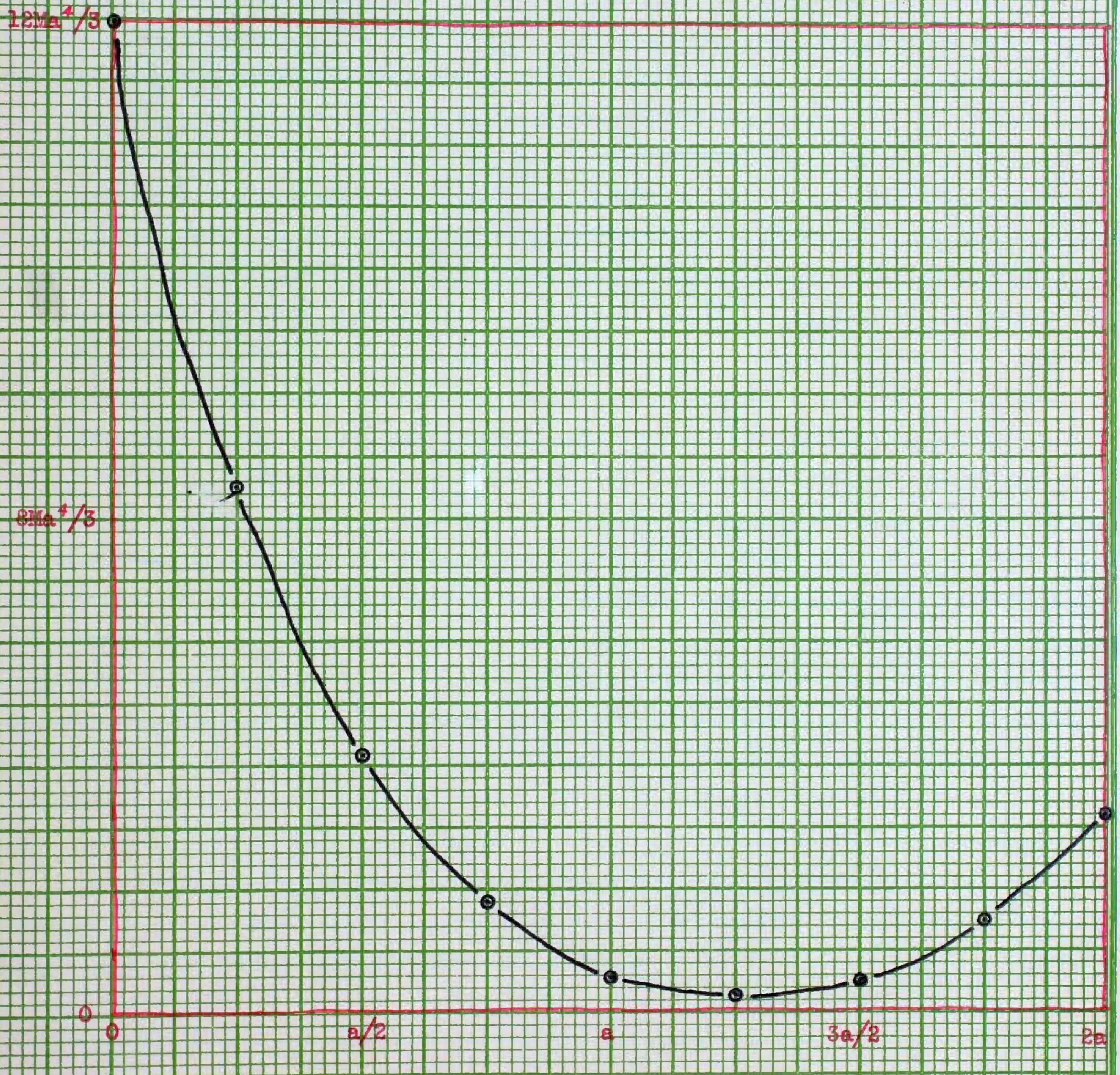


Fig. 31 The Heterogeneous Bar (σ varies as d)

VARIATION OF FOURTH MOMENTS (CONT.)

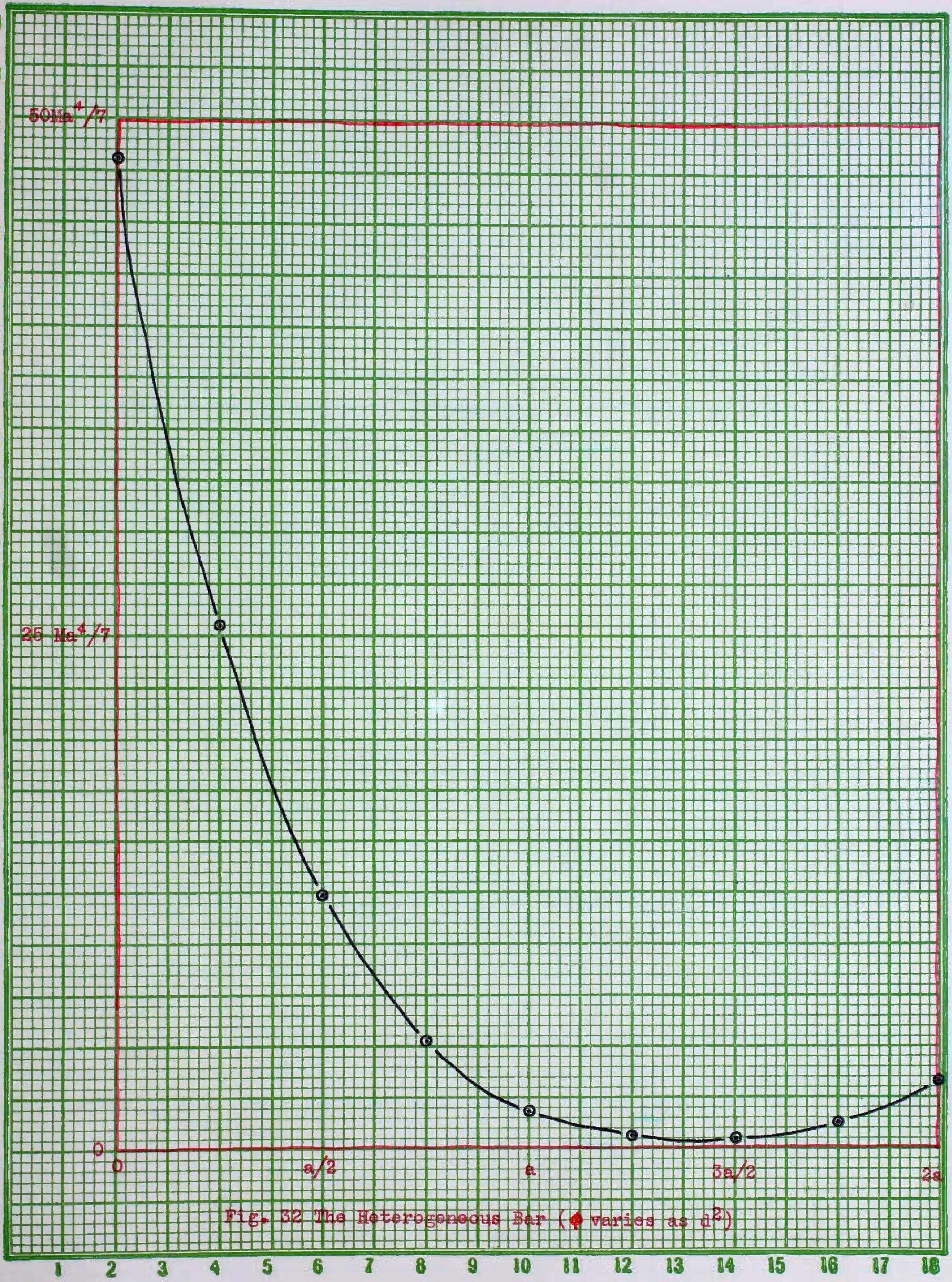


Fig. 32 The Heterogeneous Bar (ϕ varies as d^2)

VARIATION OF FOURTH MOMENTS (CONT.)

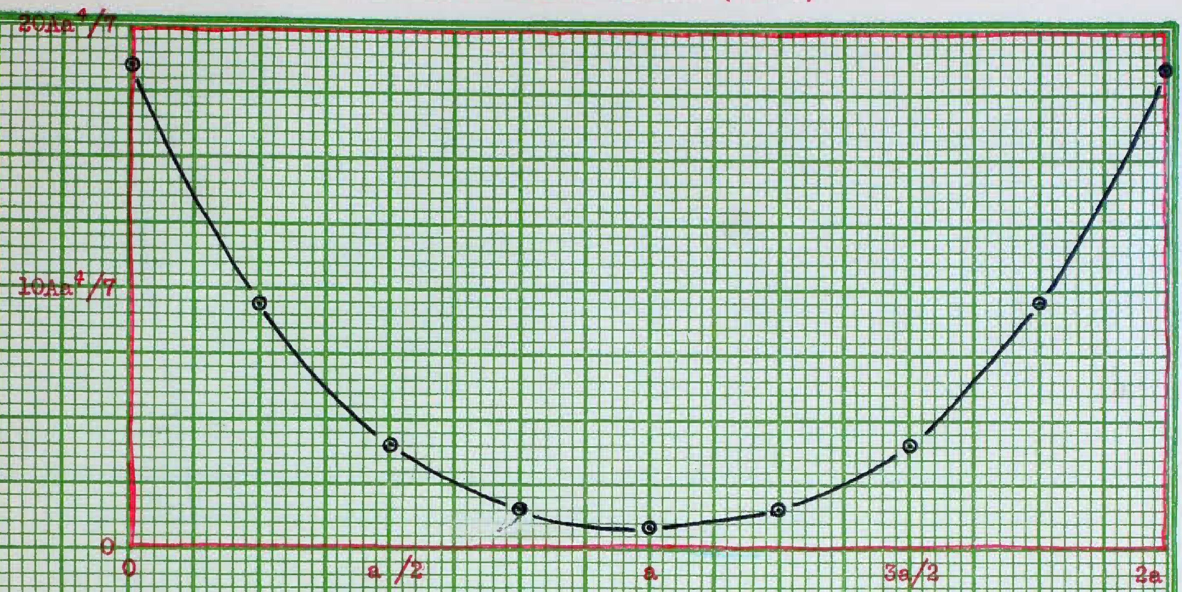


Fig. 33 The Ellipse

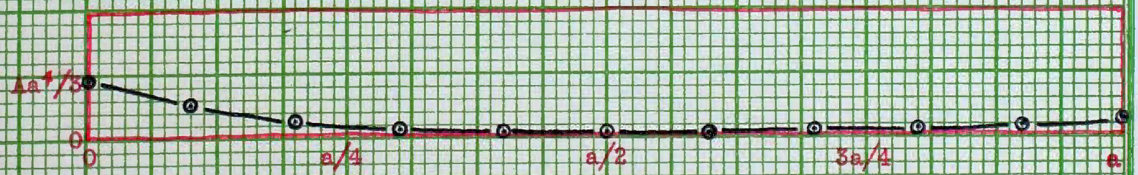


Fig. 34 The Parabola

VARIATION OF FOURTH MOMENTS (CONT.)

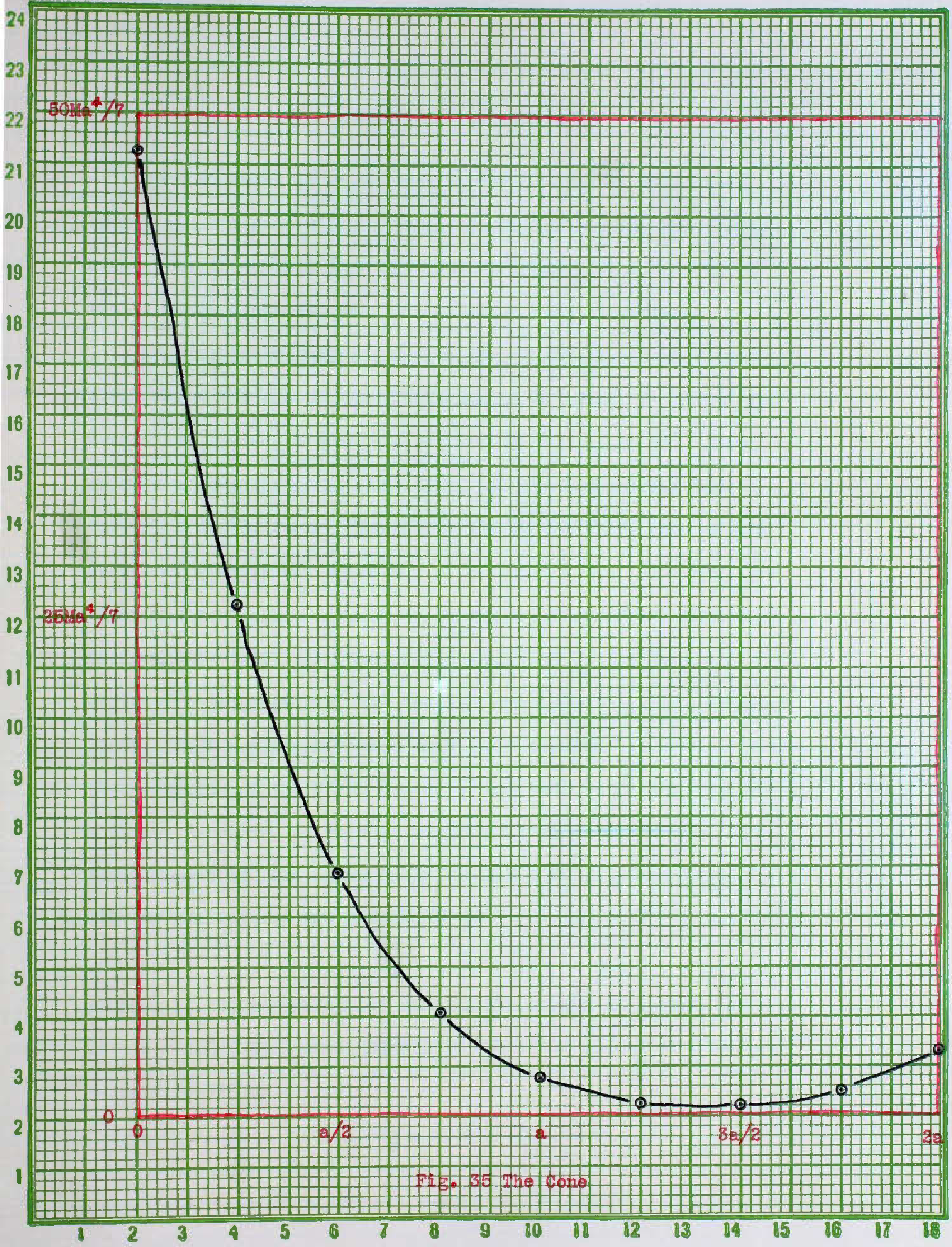


Fig. 35 The Cone

SUMMARY AND CONCLUSIONS

Moments about lines through particular points in various solids and areas were found in the different mathematical references, but in no case were the moments graphed as a function of the distance along the axis of the given body or surface.

As previously defined, the center of gravity is that point about which the sum of the first moments is zero. In all of the cases illustrated, regardless of whether the figure was homogeneous or heterogeneous, the first moments about lines equidistant from the center of gravity and on opposite sides of the center of gravity are equal but of opposite sign. Also when the lines and planes about which the moments are found are placed at uniform intervals, the moments for any given figure form an arithmetic progression. The results, when plotted, form straight lines. From a study of these graphs one is lead to conclude that the variation of the first moment of any body along any axis is a linear function. A general empirical formula for finding the first moment of mass of any body about a line or plane perpendicular to the axis of the body would be

$$I_y = M\,(\overline{x} - x),$$

where $\overline{x}$ is the perpendicular through the center of gravity and M is the mass of the body. Similarly the first moment of area may be represented by

$$I_y = A\,(\overline{x} - x),$$

where A is the area of the body. These empirical formulae were not found by the writer of this thesis in any text book that discussed moments.

Each of the graphs of the second moments show that the curves of the results are symmetrical to a line perpendicular to the axis and through the centroid. In order to make the curve of Fig. 18 completely symmetrical the moments along the axis must be found beyond the end of the bar to the point 8 a/3. Similarly to make Fig. 19, Fig. 22 and Fig. 23 symmetrical, the moments would have to be computed to the points 3a, 6 a/5, and 3a respectively. These curves are parabolas of different curvatures.

The third moments of mass or area of the homogeneous bar and the ellipse give curves that are symmetrical with respect to the centroidal point. The third moments of mass or area of all other bodies examined did not give a curve symmetrical with respect to the centroidal point even though the curves might be extended as for the second moments.

The fourth moments of mass or area of the homogeneous bar and the ellipse also give curves that are symmetrical with respect to the centroidal point, but the curves for the other bodies studied are not symmetrical.

By a comparison of Fig. 10 and Fig. 14, Fig. 19 and Fig. 23, Fig. 26 and Fig. 29, and Fig. 32 and Fig. 35 it can be seen that the cone and the bar whose mass varies as the square of the distance from one end have identical first moments, identical second moments, identical third moments and identical fourth moments. This will hold true as long as the altitude and length of the cone and bar, respectively, are equal and the moments of mass of the cone are taken about its axis, because their centroidal points will have similar locations. Naturally, the circle, a special case of the ellipse, will always give results corresponding to those of the ellipse.

BIBLIOGRAPHY

Bayley, Paul Laverne and Bidwell, Charles Clarence.

An Advanced Course in General College Physics.

New York, The Macmillan Company, 1936, p. 50-53.

Contains a discussion of moment of inertia and moments of inertia about parallel axes.

Camp, Burton Howard.

The Mathematical Part of Elementary Statistics.

Boston [etc.], D.C. Heath and Company, 1934, p. 18-35, 50-51, 135-144, 254, 319.

Shows methods of computing moments.

Delaker, Hans H. and Hortig, Henry E.

The Calculus. First Edition, second impression.

New York, McGraw-Hill Book Company, Inc., 1930, p. 178-183, 198-208.

Methods are shown for computing the radius of gyration and moment of inertia for rectangles, a quadrant of a circle, a parabola and a cone. Moments of inertia of volumes of revolution are also discussed.

Erikson, Henry A.

Elements of Mechanics. Third Edition.

New York and London, McGraw-Hill Book Company, Inc., 1936, p. 119-125.

The moments of inertia for a single particle, a hoop and a solid cylinder are demonstrated as well as the general formula for finding moments about axes parallel to the axis through the centroid of a body.

Fite, W. Benjamin.

Advanced Calculus.

New York, The Macmillan Company, 1938, p. 111-113

The moment of inertia of a plane area is illustrated.

Foley, Arthur L.

College Physics. Second Edition.

Philadelphia, P. Blakiston's Sons & Company, Inc. [c1937] p. 96-101.

The moment of inertia is defined and illustrated. Equations are given for the moments of inertia of a sphere and a cylinder and any body with axes at various distances from the center of gravity.

Harper, F. S.

Elements of Practical Statistics.

New York, The Macmillan Company, 1930, p. 151.

A footnote explains how kurtosis may be found by using the first and second moments.

Holzinger, Karl John.

Statistical Methods for Students in Education

Boston [etc.], Ginn and Company, [c1928], p. 338-346

The method of moments is applied to frequency data and fitting a normal curve.

Jeans, J. H.

An Elementary Treatise on Theoretical Mechanics.

Boston [etc.], Ginn and Company, [c1907], p. 289-290, 310-313.

This reference explains kinetic energy of rotation, coefficients of inertia, ellipsoid of inertia and principal axes of inertia.

Lindsay, Robert Bruce.

Physical Mechanics.

New York, D. Van Nostrand Company, Inc., 1933, p. 147-158, 173-179.

The general formula for moment of inertia is derived. Calculations of moment of inertia for the homogeneous bar, an ellipse and a solid body were illustrated. The method of finding the center of mass was also demonstrated.

Love, Clyde E.

Differential and Integral Calculus. Third Edition.

New York, The Macmillan Company, 1937, p. 220-229.

Contains a chapter on moments of inertia.

Miller, John Anthony and Lilly, Scott Barrett.

Analytic Mechanics.

Boston [etc.], D. C. Heath and Company, [c1915], p. 213-231.

A chapter is devoted to the moment of inertia.

Osgood, William F.

Advanced Calculus.

New York, The Macmillan Company, 1930, p. 58-64.

Finding moments and products of inertia by integration is described.

Pierce, B. O.

A Short Table of Integrals. Third revised edition.

Boston [etc.], Ginn and Company, [c1929], p. 156.

Contains a good set of integral forms.

Reynolds, Joseph Benson.

Analytic Mechanics.

New York, Prentice-Hall, Inc., 1929, p. 241-251.

Moments of inertia and radii of gyration are worked out for bars, cones and tetrahedrons.

Rietz, Henry Lewis, Editor-in-chief.

Handbook of Mathematical Statistics.

Boston [etc.], Houghton Mifflin Company, 1924, p. 15, 20, 65-70, 92-96.

The method of moments and modification of frequency moments is discussed.

Smith, Edward S.

Salkover, Meyer and Justice, Howard K. Calculus.

New York, John Wiley and Sons, Inc., 1938, p. 290-320, 495-497.

A complete discussion is given of center of gravity, centroid of volume and plane area and moments of inertia of both homogeneous and heterogeneous masses and volumes. Formulae for triple integration to find moments of inertia of solids are given.

Smith, Percy F. and Longley, William Raymond.

Theoretical Mechanics.

Boston [etc.], Ginn and Company, [c1910], p. 1-13, 21-25, 132-136, 220-225, 234-237.

The book contains discussions of moments of mass and inertia and moment of momentum.