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Asymptotic Series

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ASYMPTOTIC SERIES

being

A thesis presented to the Graduate Faculty of the Fort Hays Kansas State College in partial fulfillment of the requirements for the Degree of Master of Science

by

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Bernard Martin
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INTRODUCTION

The purpose of this thesis is to follow the development of the asymptotic series from the beginning made by Euler, in his solution of problems, to the modern applications of asymptotic series. Also to present a definition and an analysis of asymptotic series, explain some different theories of their use, and show the ways in which they find common usage.

The use of asymptotic series is a comparative new development in the field of applied mathematics. The divergent series were not used as a method of calculation until Euler showed that they gave close approximations in the solution of problems.

The study is of interest because of the number of applied uses we have for asymptotic series. These applications are used in many fields of scientific research. The ways in which an asymptotic series solution may be derived for a problem are fascinating because of the ingenuity by which they are set up.

The material was taken from works on the theory and the applied use of asymptotic series, and from correspondence with the Engineering Department of the General Electric Company, Schenectady, New York.
CHAPTER I

History of Asymptotic Series

Before the time of Euler (1707-1783), there had been little done with the divergent series as a method of approximation. However, there had been some use of series in the approximation of different mathematical relations. It was in the computation of " by Vieta that an infinite series product was used in 1593. During the seventeenth century James Gregory had used series computation for the tan x, sec x, and arc tan x. Gregory also receives the credit of having originated the idea of expressing a logarithm by means of a series. The antitrigonometric series for arc sin x had been given by Newton in 1669.¹

Euler, by his labors, did more than anyone else up to his time to give to analysis its generality and symmetry. Generality and symmetry are the qualities recognized as belonging to modern analysis. He worked out the theory of the rotation of a body around a fixed point, established the general equation of motion of a free body, and the general equation of hydrodynamics. The writer is most

interested in the approximate solutions that Euler used. He used the approximate solution on the "problem of three bodies" and by using the method of approximation was one of the first to solve with success the theory of the moon's motion. The subject of infinite series received new life under him. Through his researches on series we owe the theory of definite integrals by the development of the Eulerian integrals. He warned his readers against the use of divergent series but was careless himself, probably due to the fact non convergent series of various types give close approximations. He was one of the first to use the asymptotic series in the solution of problems.¹

In the early part of the nineteenth century divergent series were thought to be useless by a number of mathematicians of that time; the belief was that only convergent series could be used in the solution of problems.²

In 1822 Fourier (1768-1830) published his work entitled "La Théorie de la Chaleur". This book contained Fourier's Series in the same form as they are now. This work contains methods of solving problems in heat and wave vibration which are used to a great extent today.

² Ibid. p. 391.
This book brought to a close a long controversy and proved that any graphically given function of a real variable can be represented by a trigonometric series. The weak point in Fourier's analysis lies in his failure to prove generally that the trigonometric series actually converges to the value of the function.\textsuperscript{1}

Poisson (1781-1840) made himself distinguished by his work on applied mathematics and physics. He wrote four hundred volumes, most of them on applied mathematics. His memoirs in pure mathematics of most importance are those on definite integrals and series and their application to physical problems. His work on electrostatics and magnetism originated a new field of mathematical physics. He was considered one of the leading analysts of his time, and there are a great number of problems on elasticity to which he has contributed. He was the originator of many new methods for the solution of problems. He also differed from others by his methods of solution. He did not use the method of the definite integral but instead used the method of finite summations.\textsuperscript{2}

Bessel (1784-1846) introduced into pure mathematics functions which are now known by his name; these were introduced in 1824. Their use had appeared before with

\begin{enumerate}
\item Cajori, \textit{A History of Mathematics}, p. 270.
\item Ibid. p. 456.
\end{enumerate}
different variations during the time of Euler. Bessel used these functions in the solution of problems of vibration. These functions which he called $J_n(x)$ are very useful today in the solution of problems in applied mathematics. They take the form of an infinite series, which gives asymptotic solutions for different values of the variable. These asymptotic series were convenient to use as Bessel had worked out tables for their solution. In modern books on applied mathematics there are tables for the different Bessel's functions which gives the function for many values of the variable.\(^1\)

A. L. Cauchy (1789-1857) wrote his Analyse Algebrique in 1821 which contained a rigorous treatment of series and set up definite rules of testing for convergent and divergent series. He used the ratio test for the convergence of a series. He reintroduced the concept of an integral of a function as the limit of a sum. He also did work in physics on astronomy, light and elasticity.\(^2\)

Stokes (1819-1903) distinguished himself in the field of applied mathematics. His memoir published on "Friction of Fluids in Motion" in 1845 contained an

\(^1\) Cajori, A History of Mathematics, p. 448.
\(^2\) Ibid. p, 359, 373.
analysis of the motion of a medium near any point. He analyzed it as containing three constituents; a motion of pure translation, one of pure strain and one of pure rotation. In using his results on viscous fluids, he was led to the use of general dynamical equations. The two elastic constants in the equation for the elastic solid he considered to be independent and not reducable to one as is the case in Poisson's theory. In 1847 Stokes examined anew the theory of oscillatory waves.1

Poincare (1854-1912) worked out a powerful method of solving differential equations by the use of asymptotic series. He has proved that every linear differential equation which has polynomial coefficients may be solved by asymptotic series. He was not the type to spend his time on the history of a certain condition but instead strove for methods of solution in the problems that he encountered.2 This was probably the reason that he developed his different solutions.

The use of asymptotic series is general in several fields of applied mathematics today. Asymptotic series are used in the fields of radio, electricity, sound, etc. The commercial usage has meant greater development of asymptotic series.

2. Ibid. p. 389.
CHAPTER II

Properties of Asymptotic Series

Asymptotic series in general use are of the non-convergent type, where the terms begin to decrease, and reach a minimum, afterwards increasing. If the sum is calculated to a stage at which the terms are sufficiently small, it is possible to obtain an approximation with a degree of accuracy represented by the last term retained. This is the case with many series which are convenient for numerical calculations.¹

An asymptotic series is a series that gives a close approximation to the function that is under consideration. Starting from some function \( J(x) \), it is developed formally in a series

\[
a_0 + a_1/x + a_2/x^2 + a_3/x^3 + \ldots
\]

This series is not convergent, but the sum of the first \((n+1)\) terms gives an approximation to \( J(x) \) which differs from \( J(x) \) by less than \( K_n/x^{n+1} \), where \( K_n \) depends only on \( n \) and not on \( x \). Let \( S_n \) denote the sum of the first \((n+1)\) terms and the

\[ \lim_{n \to \infty} x^n (J - S_n) = 0. \]

In all such cases as the above, the series is asymptotic to the function.¹

Two functions, \( f(x) \) and \( F(x) \), which are so related that for a particular endless progress of the independent variable \( x \), their ratio approaches one, or what amounts to the same thing \( (f(x) - F(x))/f(x) \) approaches 0, the functions are said to be asymptotic approximations, each to the other. Since both of the functions are infinite, neither can be a constant. The following formula illustrates the relation of asymptotic approximations,

\[
f(x) - F(x) = e f(x) \quad \text{or} \quad F(x) = (1 - e) f(x),
\]

where \( e \) is an infinitesimal. Two functions that are asymptotic approximations of each other must be infinites of the same order, and their difference of approximation, must be infinitesimal, or finite in the sense of being neither positive or negative infinity, or infinite of lower order than the functions themselves. The difference ratio of the two functions

\[
( f(x) - F(x) )/f(x)
\]

must be an infinitesimal, and the approximation is more or less close according as the infinitesimal order of

difference ratio is higher or lower degree. The distinction between infinitesimal and asymptotic approximation is that the former applies to infinitesimal and finite functions; the latter, to infinites. The relationship of approximation is fundamentally the same in the two cases. If \( f(x)/F(x) \) approaches \( A \), where \( A \) is a constant different from 0, \( A \cdot F(x) \) is an asymptotic approximation to \( f(x) \).\(^1\)

Asymptotic series used for approximation are of two types, one represents the given function \( f(x) \) asymptotically for small positive values of \( x \), the other is said to represent the given function \( f(x) \) asymptotically for large positive values of \( x \).

Let the function \( f(x) \) in the interval \((0, b)\), be represented by the power series,

\[
(1) \quad A_0 + A_1x + A_2x^2 + \cdots + A_nx^n + \cdots.
\]

Assume that the limit of the sum of the series as \( x \) approaches zero is, \( f(x) = A_0 \), also that the limit as \( x \) approaches 0 is,

\[
(f(x) - A_0 - A_1x - \cdots - A_{n-1}x^{n-1})/x^n = A_n.
\]

If series (1) is divergent, the sum can no longer be spoken of in the ordinary sense, but it can be said that

\(^1\) Bromwich, Theory of Infinite Series, p. 116.
for a finite number of terms the series gives an approximation of $x$ for small values of $x$. It should be noted that the smaller $x$ is taken, the larger $n$ must be chosen in order to get the best approximation to the corresponding values of $x$.

For large values of $x$, $f(x)$ is said to be represented asymptotically by the series,

$$A_0 + A_1/x + A_2/x^2 + \ldots + A_n/x^n + \ldots.$$  

As $x$ approaches infinity, the following condition is true,

$$x^n(f(x) - A_0 - A_1/x - \ldots - A_n/x^n) = 0.$$  

The relation of a function $f(x)$, to an asymptotic series is written,

$$f(x) \sim A_0 + A_1/x + A_2/x^2 + \ldots + A_n/x^n + \ldots.$$  

The series may be convergent or divergent for the large positive values of $x$. In the case of a convergent series the actual value of $f(x)$ is obtained when $n$ is allowed to become infinite. The best approximation of $f(x)$ by an asymptotic series is obtained if the series is terminated with the term having the smallest absolute value. If the series is of the form $S A_n x^n$, the number of terms increases as $x$ takes smaller values. If it is of the form $S A_n/x^n$, to get the best approximations, $x$ must take larger values to increase the number of terms.\(^1\)

\(^1\) Townsend, Functions of Real Variables, p. 388, 389
Asymptotic series are obtained from a given integral by three methods,

(1) Integration by parts

(2) Use of symbolic operators

(3) Expansion of some function in the integral

If the method of integration by parts is applied to the error-function integral, defined by the equation,

\[ \text{erf } x = \int_0^x e^{-t^2} dt. \]

When \( x \) is small a series suitable for calculation is deduced by expanding the exponential and integrating term by term. If the integral is used,

\[ u = \int_{x^2}^{\infty} e^{-t^2} dt = \sqrt{\pi} - \text{erf } x, \]

by writing \( t^2 = s \), an asymptotic series for the integral is readily found, then

\[ u = \int_x^{\infty} e^{-s} \frac{ds}{2\sqrt{s}} \]

The last integral is integrated by parts, which gives

\[
\begin{align*}
    u &= \left[-e^{-s}/2s\right]_{x^2}^{\infty} - \int_{x^2}^{\infty} e^{-s}/(2s^{3/2}) \, ds \\
    &= e^{-x^2}(1/2x) - \left[e^{-s}/(2s^{3/2})\right]_{x^2}^{\infty} - \int_{x^2}^{\infty} 1\cdot 3e^{-s}/(2^3s^{5/2}) \, ds, \\
    &= e^{-x^2}(1/2x - 1/2x^3 - 1\cdot 3/2x^5 - 1\cdot 3\cdot 5/2^4x^7)
\end{align*}
\]
\[- \int_{x^2}^{\infty} 1 \cdot 3 \cdot 5 \cdot 7 \cdot e^{-s}/(2^5 s^{9/2}) \, ds.\]

The last line is obtained from the preceding by two integrations and it is possible to continue this process indefinitely. The further the method is followed the smaller the error. The remainder in the last formula is less than the next term in the series, which is

\[1 \cdot 3 \cdot 5 \cdot 7 / (2^5 x^9) \int_{x^2}^{\infty} e^{-s} \, ds = 1 \cdot 3 \cdot 5 \cdot 7 / (2^5 x^9) \, e^{-x^2}.\]

The error by stopping at any term in the asymptotic series is less than the following term in the series. ¹

If the symbolic method is used on the error-function integral, an asymptotic series is more easily obtained. If \(1/D\) denotes the operation of integration from \(x^2\) to infinity, then

\[e^{-s}/(D \cdot 2 \sqrt{s}) = e^{-s}/2(D - 1)\sqrt{s}\]

\[= - e^{-s}/2 \left(1 + D + D^2 + D^3 + \ldots\right) 1/\sqrt{s}\]

\[= (- e^{-s}/2)(1/\sqrt{s} - 1/(2 \cdot s^{3/2}) + 1.3/(2^2 s^{5/2}) \ldots).\]

If \(s = x^2\) is the lower limit of the integral, the asymptotic series is obtained,

\[u = e^{-x^2}(1/2x - 1/2^2 x^3 + 1.3/2^3 x^5 - \ldots).\]

If the remainder formula is used,

\[1/(D-1) = -(1 + D + D^2 + \ldots + D^{n-1}) + D^n/(D-1).\]

¹ Bromwich, Theory of Infinite Series, p. 332.
It is easy to see that the same remainder-integral is left as in the method of integration by parts. The same inference in regard to the magnitude of the remainder is formed.\textsuperscript{1}

The method of expansion may be written as follows,

\[ s = x^2 + v \]

and

\[ u = e^{-x^2} \int_0^\infty e^{-v} dv/(2 \sqrt{x^2 - v}). \]

The following asymptotic series is found from the expansion,

\[ 1/\sqrt{x^2 + v} = 1/x - 1 \cdot v/2x^3 + 1 \cdot 3 \cdot v^2/2 \cdot 4 \cdot x^5 \ldots \]

The remainder at any stage is less than the following term. The integral

\[ \int_0^\infty v^n e^{-v} dv = n! \]

and the same results are obtained for \( u \) and its asymptotic series again.\textsuperscript{2}

Asymptotic series are like convergent series in that they can be added, subtracted and multiplied and asymptotic expansions of the given functions obtained. Theoretically, the convergent series can be pushed to an arbitrary degree of approximation, while an asymptotic series cannot. In practice an asymptotic series usually gives a better degree of approximation for

\textsuperscript{1} Bromwich, Theory of Infinite Series, p. 333.
\textsuperscript{2} Ibid. p. 333.
numerical work than a convergent series because the value of an asymptotic series can be found to a certain number of terms and the error calculated while convergent series must be summed to the n-th term. If some function \( f(x) \) is developed formally in an asymptotic series,

\[
f(x) \sim A_0 + A_1/x + A_2/x^2 + \ldots
\]

The series is asymptotic to the function and the sum of the first \((n+1)\) terms gives an approximation to \( f(x) \).

It follows from the definition of asymptotic series that they can be added and subtracted.\(^1\)

The proof that asymptotic series can be multiplied will follow. Let \( J(x) \cdot K(x) \) be represented as follows,

\[
J(x) \sim A_0 + A_1/x + A_2/x^2 + \ldots
\]

\[
K(x) \sim B_0 + B_1/x + B_2/x^2 + \ldots
\]

and the formal product of the two represented by the single series,

\[
I(x) \sim C_0 + C_1/x + C_2/x^2 + \ldots
\]

If the sums of the first \((n+1)\) terms of the three series are considered and those sums are represented in the above order by \( S_n \), \( T_n \), and \( H_n \). The asymptotic relationship may be written thus,

---

where \( p, a \) are functions of \( x \) which tend to zero as \( x \) approaches infinity. \( H_n \) coincides with the product \( S_n \cdot T_n \) up to and including the terms in \( 1/x^n \). The following relationship exists,

\[
S_n \cdot T_n = H_n + P_n/x^{2n},
\]

where \( P_n \) is a polynomial in \( x \), whose highest term is of the (\( n-1 \))-th degree. As \( x \) approaches infinity,

\[
J(x) \to A_0, \quad K(x) \to B_0, \quad p \to 0, \quad a \to 0, \quad \text{or} \quad (J(x) \cdot K(x) - H_n) = 0.
\]

Thus the product \( J(x) \cdot K(x) \) is represented asymptotically by \( I(x) \), or asymptotic series can be multiplied together as if they were convergent, and any power of an asymptotic series may be obtained.\(^1\)

An asymptotic series can be integrated term by term, thus obtaining an asymptotic expansion of the integral of the function, provided the first two terms of the expansion are zero and the integral is taken over the interval \((x, \infty)\). The following gives the integration of an asymptotic series (in which \( A_0 = 0, A_1 = 0 \)).

\[
J(x) \sim \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x^4} + \ldots
\]

---

and \[ |J - S_n| < \frac{e}{x^n}, \text{ if } x > x_0. \]

The integral of the above from \( x \) to infinity is,
\[
\left| \int_x^\infty J \, dx - \int_x^\infty S_n \, dx \right| < \frac{e}{(n-1)x^{n-1}}.
\]

The integration from \( x \) to infinity of the asymptotic series \( J(x) \) is represented asymptotically by,
\[
A_2/x + A_3/2x^2 + A_4/3x^3 + \cdots.
\]

An asymptotic series cannot safely be differentiated without investigation, for the existence of an asymptotic series for \( J(x) \) does not imply the existence of one for \( J'(x) \). If it is known that \( J'(x) \) has an asymptotic expansion, it must be the series obtained by the ordinary rule for term-by-term differentiation. If the asymptotic series is,
\[
J(x) \sim A_0 + A_1/x + A_2/x^2 + \cdots,
\]
then
\[
\lim x^{n+1}( J(x) - S_n(x) ) = A_{n+1}.
\]

Thus the differential coefficient
\[
x^{n+1}( J'(x) - S_n(x) ) = (n+1)x^n( J(x) - S_n(x) ),
\]
if it has a definite limit, must tend to zero. But
\[
x^n( J(x) - S_n(x) ),
\]
does tend to zero, so that the limit of,
\[
x^{n+1}( J'(x) - J(x) ),
\]

---

if it exists, is zero. If \( J'(x) \) has an asymptotic series, it is
\[
- \frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots
\]

The following will deal with the way in which asymptotic series present themselves in the solution of differential equations. If the following differential equation is considered,
\[
\frac{dy}{dx} = \frac{a}{x} + by, \quad (b > 0),
\]
by means of an asymptotic series
\[
y = A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots
\]
on substitution this gives,
\[
- \frac{A_1}{x^2} - \frac{2A_2}{x^3} - \frac{3A_3}{x^4} - \ldots = \frac{a}{x} + b(A_0 + \frac{A_1}{x} + \frac{A_2}{x^2} + \ldots).
\]
Then
\[
A_0 = 0, \quad A_1 = -\frac{a}{b}, \quad A_2 = -\frac{A_1}{b} = \frac{a}{b^2},
\]
\[
A_3 = -\frac{2A_2}{b} = -\frac{2a}{b^3}, \text{ etc.}
\]
Thus the formal solution is found to be
\[
y = -\frac{a}{bx} \left(1 - \frac{1}{bx} + 1 \cdot 2 \left(\frac{1}{bx}\right)^2 - \ldots\right),
\]
and this represents the integral
\[
- a \int_0^\infty \frac{e^{-t}}{t - bx} \, dt,
\]
and it is now easy to verify directly that this integral does satisfy the given equation.

2. Ibid. p. 349.
CHAPTER III

Some Theories of Asymptotic Series

This chapter will discuss some of the various theories that employ asymptotic series as a method of solving problems. The general type of solution will be taken up first in each case and in some instances the particular solution will follow. There are so many particular types that can be derived from the general form that it would be impossible to mention all of them.

In asymptotic series usage the general type of solution is usually considered and the particular type of solution is obtained from the general. The particular solution is set up from the general to fit the conditions of the problem under consideration.

Euler's formula of summation will be considered first, as he was one of the first to use the asymptotic series solution of problems. He used the method of summation to calculate certain finite sums. By setting the function,

\[ f(x) = \frac{1}{x}, \quad x = n, \]

the series that follows is

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} = \log n + \frac{1}{2n} - \frac{B_1}{2n^2} + \frac{B_2}{24n^4} - \cdots + c. \]
If this series is continued to infinity it does not converge because,

$$\frac{B_r}{B_{r-1}} = \frac{2r(2r-1)}{4r^2} \cdot \frac{S(1/n^{2r})}{S(1/n^{2r-2})}.$$ 

If \( r \) is greater than 3,

$$S(1/2^{r-2}) < \left( \frac{1}{1-1/2^4} \right),$$

and

$$S(1/n^{2r}) > 1.$$ 

This makes,

$$\frac{(B_r/2rn^{2r})/(B_{r-1}/2(r-1)n^{2r-2})}{15(r-1)^2/16n^2r^2} >$$

The terms in the series steadily increase in numerical value of \( r \) (depending on \( n \) and greater than the next integer greater than \( 1-nr \)). Euler knew that this series does not converge for \( n = 1 \), and he employed it for \( n \) equal to 10 to calculate his constant,

$$c = .5772156649015328(6060) \quad \ldots$$

this he regarded as the "sum" of the series,

$$\frac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} \quad \ldots$$

The error of calculation by stopping at any term is less than the next term in the series, therefore this series can be used for numerical work even though it is not convergent. ¹

If Euler's formula is applied to Stirling's series

and the general formula is taken as, \( f(x) = \log x \), it gives,

\[
(1) \log (n)! = \int_{1}^{n} \log x \, dx + \frac{1}{2} \log n + B_{1}/2n - B_{2}/3 \cdot 4n^{3} + \ldots + c.
\]

Here the error at each stage is numerically less than the next term, because \( r^{2n}(x) \) is negative (for all positive values of \( x \)).

If (1) is integrated, the result is

\[
\log (n)! = c_{1} - \frac{(n+1/2) \log n - n + B_{1}/1 \cdot 2n - B_{2}/3 \cdot 4n^{3}}{2}.
\]

To find the constant \( c_{1} \), Wallis's theorem of sine is used, this gives

\[
\frac{\pi}{2} = 2/1 \cdot 2/3 \cdot 4/3 \cdot 4/5 \cdot \ldots = \lim_{n \to \infty} \left( \frac{2^{n} \cdot n!}{(2n)!} \right)^{2} (2n-1).
\]

\[
\log (1/2\pi) = \lim_{n \to \infty} \left( \frac{4n \log 2 + 4 \log (n)! - 2 \log (2n)! - \log (2n)}{n} \right).
\]

The general formula gives,

\[
2 \log (n!) - \log (2n)! = c_{1} + \frac{1}{2} \log n - (2n + 1/2) \log 2 + Q (1/n).
\]

Thus the \( \log (1/2\pi) \) is,

\[
= \lim_{n \to \infty} \left( 4n \log 2 + 2c_{1} + \log n - 2 \log (2n)! - \log (2n) \right).
\]
(4n+1) log 2 - log (2n) )

= 2c_1 - 2 \log 2,

thus \( c_1 = \frac{1}{2} \log 2\pi \).

By the proper substitutions the preceding gives Stirling's series

\[
\log (n!) = (n+1/2) \log n - n + 1/2 \log (2\pi) + \frac{B_1}{2n} - \frac{B_2}{3 \cdot 4n^3} + \ldots
\]

in which \( n \) denotes a positive integer.

If Euler's summation formula is used in the interval from \( x \) to \( x+n \), the series for \( \log ( T(1+x) ) \) is found.

\[
\log \left( \frac{x(1+x)}{(n+x)(1+x/n)} \right) = \int_x^{x+n} \log e \, de + 1/2 \left( \log x + \log (x+n) \right) - \frac{B_1}{2x} + \frac{B_2}{(3 \cdot 4n^3)} - \ldots + Q(1/n).
\]

If this is subtracted from Stirling's series for \( \log (n!) \), it gives

\[
- \log \left( \frac{(1+x/1)(1+x/2)}{(1+x/n)} \right) = \int_x^{x+n} \log e \, de + \int_0^n \log e \, de + 1/2 \log x + 1/2 \log (2\pi) + \frac{B_1}{2x} - \frac{B_2}{(3 \cdot 4x^3)} + \ldots + Q(1/n).
\]

---

By putting \( e = n + N \), the difference of the two integrals in the last formula is equal to,

\[
x \log x - x - x \log n - \frac{Q(1/n)}{N}.
\]

Thus it is found,

\[
x \log n - \log \left( \frac{(1+x/1)(1+x/2)\ldots(1+x/n)}{x} \right) = (x+1/2) \log x - x + \frac{1}{2} \log (2\pi) + \frac{B_1}{2x} - \frac{B_2}{(3 \cdot 4x^3)} + \frac{Q(1/n)}{N}.
\]

When \( n \) approaches infinity the left side of the series equation tends to \( \log (T(1+x)) \) and the result is

\[
\log (T(1+x)) = (x+1/2) \log x - x + \frac{1}{2} \log (2\pi) + \frac{B_1}{2x} - \frac{B_2}{(3 \cdot 4x^3)} + \frac{Q(1/n)}{N}.
\]

This series is exactly the same form as the series originally found for \( \log (n!) \).\(^1\)

The preceding is a form of the Gamma function which is,

\[
T(n) = \int_0^\infty x^{n-1} e^{-x} \, dx, \quad (n > 0),
\]

\[
T(1) = \int_0^\infty e^{-x} \, dx = 1.
\]

If the above integral is integrated by parts it gives,

\[
\int_0^\infty x^n e^{-x} \, dx = \left[ -x^n e^{-x} \right]_0^\infty - n \int_0^\infty x^{n-1} e^{-x} \, dx = n \int_0^\infty x^{n-1} e^{-x} \, dx, \text{ and } T(n+1) = nT(n).
\]

---

If the value of $T(n)$ is known for $n$ between any two successive integers, the value of $T(n)$ may be found by the preceding for any positive $n$. Tables for $\log T(n)$ for small values of $n$ have been computed for use in applied problems.\\(1\\)

Stokes has given an asymptotic expression for the series,

$$S(\frac{T(n+a_1+1)}{T(n+b_1+1)} - \frac{T(n+a_2+1)}{T(n+b_2+1)}) x^n = S(x^n),$$

where $x$ is real and $s > r$. The following substitutions are made in the summation, $s-r = u$, $S(b) - S(a) = R$, and the $x_{t+p}$ term is considered, where $t$ is large, and $p$ is not of higher order than $\sqrt{t}$. If terms of order $1/\sqrt{t}$ are neglected, the following is derived from Stirling's series,

$$\log x_{t+p} = (t+p) \log x - u \left( (t+1/2) \log t + \frac{1}{2} \log (2\pi) - t \right) - (pu + R) \log t - u p^2 / 2t.$$

It is most convenient to suppose that $x$ is of the form $\frac{u}{t}$, where $t$ is an integer; and then $X_t$ is the greatest term because $\log x = u \log t$, so that the terms of the first degree in $p$ cancel. This gives

---

\[ \log X_{t+p} = ut - \frac{1}{2}u \log (2\pi t) - R \log t - \frac{1}{2}up^2/t \text{ or,} \]
\[ X_{t+p} = \left( e^{tu} t^{-R} \right)/(2\pi t)^\frac{3u}{2} \exp (-1\cdot up^2/2t). \]

If \( X_{t+p} \) and \( X_{t-p} \) are combined, this gives the asymptotic expression,
\[ \left( e^{tu} t^{-R} \right)/(2\pi t)^\frac{3u}{2} (1 + 2q^4 + 2q^9 + \ldots), \]
where \( q = e^{-\frac{2u}{t}}. \)

It follows from Abel's theorem since \( q \) approaches the limit one, the series in brackets is represented asymptotically by \( \pi^\frac{3}{2}(1-q)^{-\frac{3}{2}} \), or by \( (2\pi t/u)^{\frac{3}{2}} \).

The asymptotic expression for the series is,
\[ \left( e^{tu} t^{-R} \right)/u^{\frac{3}{2}}(2\pi t)^{\frac{3}{2}}(u-1) \text{, where } t = x^{1/u}. \]

The solution of Bessel's equation will be taken into consideration where the variables are of an integer form. Bessel's equation is,
\[ \frac{d^2z}{dx^2} + \left( \frac{1}{x} \right) \frac{dz}{dx} + \left( 1 - m^2/x^2 \right) z = 0, \]
and if \( m = 0 \), it reduces to
\[ (1) \frac{d^2z}{dx^2} + \left( \frac{1}{x} \right) \frac{dz}{dx} + z = 0. \]
Assume that \( z \) can be expressed in whole powers of \( x \), and let \( z = S (a_n x^n) \). Substitute this value in (1)

and it gives,

\[ S(n(n-1)a_n x^{n-2} + na_n x^{n-2} + a_n x^n) = 0, \]

an equation which must be true no matter what the value of \( x \). The coefficients of any given power of \( x \), as \( x^{k-2} \), must vanish, and

\[ k(k-1)a_k + ka_k + a_{k-2} = 0, \]
\[ k^2a_k + a_{k-2} = 0, \]

whence we obtain

\[ a_{k-2} = -k^2a_k. \]

This is the only relation that need be satisfied by the coefficients in order that \( z = S a_k x^{k} \) shall be a solution of (1). If

\[ k = 0, a_{k-2} = 0, a_{k-4} = 0, \text{ etc.} \]

The beginning is with \( k = 0 \) as the lowest subscript and the following subscripts are found from the above.

\[ a_k = -\left( a_{k-2} \right)/k^2 \]
\[ a_2 = -a_0/2^2 \]
\[ a_4 = a_0/(2^2 4^2) \]
\[ a_6 = -a_0/(2^2 4^2 6^2), \text{ etc.} \]

If these values are placed in the power series that was given as a solution of \( z \), it gives the solution.
This is a solution of (1) where \( a_0 \) may be taken as any integer, provided the series is convergent.

Take \( a_0 = 1 \), and then \( z = J_0(x) \) where

\[
J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4 2^4} - \frac{x^6}{2^6 2^4 2^6} + \cdots
\]

is a solution of (1). \( J_0(x) \) is called a Bessel's Function of the zero-th order, or a Cylindrical Harmonic. It can be shown that \( J_0(x) \) has an infinite number of real positive roots, any one of which can be obtained to any required degree of approximation.

If the original Bessel's equation is used,

\[
d^2z/dx^2 + \frac{1}{x}(dz/dx) + z = 0,
\]

and \( z = x^n v \), it gives

\[
d^2v/dx^2 + \frac{(2m+1)}{x} (dv/dx) + v = 0
\]

to determine \( v \). Let \( v = S a_n x^n \), and substitute in the above.

\[
S \left(n(2m+n)a_n x^{n-2} + a_n x^n\right) = 0;
\]

whence \( a_{n-2} = -n(2m+n)a_n \).

Begin with \( n = 0 \), then \( a_{n-2} = 0, a_{n-4} = 0, \) etc, hence the set of values,

\[
a_2 = -a_0/(2^2 (m+1))
\]
Substituting these values of the coefficients in the series given for \( z \), gives

\[
z = a_0 x^m \left( 1 - \frac{x^2}{2^2(m+1)} + \frac{x^4}{2^42!(m+1)(m+2)} \right) - \frac{x^6}{2^63!(m-1)(m-2)(m-3)} + \ldots
\]

This is a solution of Bessel's Equation; \( a_0 \) is usually taken as \( \frac{1}{2^m m!} \) if \( m \) is a positive integer, and the second member of the solution is represented by \( J_m(x) \) and is called a Bessel's Function of the \( m \)-th order.

The use of Fourier's series in the solving of problems requiring approximation is of great importance, therefore Fourier's series will be considered briefly.

Series of the form involving only sines and cosines of integral multiples of \( x \) are known as Fourier's series. Examples are,

\[
b_0 + b_1 \cos x + b_2 \cos 2x + \ldots
\]

\[
a_0 + a_1 \sin x + a_2 \sin 2x + \ldots
\]

Any function of \( x \) which is single valued, finite,

---

and continuous between $x = 0$, and $x = \pi$, or if discontinuous has only finite discontinuities each of which is preceded and succeeded by continuous portions, can be developed into a series of the form.

$$f(x) = a_1 \sin x + a_2 \sin 2x + \ldots$$

where

$$a_m = \frac{2}{\pi} \int_0^\pi f(x) \sin mx \, dx = \frac{2}{\pi} \int_0^\pi f(a) \sin ma \, da,$$

and a like series of cosines gives,

$$f(x) = \frac{1}{2}b_0 + b_1 \cos x + b_2 \cos 2x + \ldots$$

where

$$b_m = \frac{2}{\pi} \int_0^\pi f(x) \cos mx \, dx = \frac{2}{\pi} \int_0^\pi f(a) \cos ma \, da.$$

The series and the function will be identical for all values of $x$ between $x = 0$, and $x = \pi$, not including value $x = 0$, and $x = \pi$, unless the given function is equal to zero for these values. At a point of finite discontinuity the series has a value equal to half the value the function has if it approaches the point of discontinuity from opposite sides. A curve represented by a sine and cosine series need not follow the same mathematical law throughout its length, but may be made up of different curves, a broken line or a locus consisting of finite parts of several different and disconnected straight lines can be represented.
If a series is wanted with values different from \( x = -\pi \), and \( x = \pi \), a new variable \( c \) is introduced. \( f(x) \) is developed into a trigonometric series for all values between \( x = c \) and \( x = -c \), by using the notation

\[
z = \left(\frac{\pi}{c}\right) \cdot x
\]

\( z = -\pi \), when \( x = -c \),

\( z = \pi \), when \( x = c \) and

\( f(x) = f\left(\frac{c}{\pi}z\right) \) can be developed in terms of \( z \). It is represented by,

\[
b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right) \cos mz \, dz
\]

\[
a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{c}{\pi}z\right) \sin mz \, dz.
\]

In the following the graph of a Fourier series will be drawn showing the approximation curves for a few values of \( x \). In the figure the curve \( y = \) the series, and the approximation in question are drawn in continuous lines, and the preceding curve and the curve corresponding to the term to be added are drawn in dotted lines. The curve for the series which follows will be drawn.

\[
y = \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \frac{1}{7} \sin 7x + \ldots
\]

1. Byerly, *Fourier's Series*, Chapter II.
2. Ibid., Chapter III.
This content will soon be analyzed to determine the nature and relevance of the figures and equations presented. The diagrams illustrate concepts related to the analysis of wave phenomena. For a detailed understanding, please refer to the corresponding sections in the text.
CHAPTER IV

Application of Asymptotic Series

This chapter will deal with some problems solved by the use of asymptotic series. The problems help to illustrate the uses to which asymptotic series can be put. The first group of problems are applications of Fourier's series. The second group require a special type of series in their solution, namely Bessel's series. The third group includes problems solved by series of other kinds. There is also a discussion of the use of series in solving equations such as Legendre's Coefficients or Zonal Harmonics, Laplace's Coefficients or Spherical Harmonics, and Lamé Functions or Ellipsoidal Harmonics.

The solution of Fourier's problem of the permanent state of temperature in a thin rectangular plate of breadth $W$ will be considered. If the two long edges of the plate are kept at the constant temperature zero, one of the short edges, called the base of the plate, is kept at the temperature unity, and the temperature of the points in the plate decrease indefinitely as the distance from the base is increased. The temperature is to be found at any point of the plate.
The equation in the Analytical Theory of Heat for the change of temperature of any solid due to the flow of heat within the solid is,

$$D_t u = a^2 \left( \frac{D^2}{x^2} + \frac{D^2}{y^2} + \frac{D^2}{z^2} \right),$$

where $u$ represents the temperature at any point of the solid and $t$ the time. In the problem of the permanent state of temperature in a thin rectangular plate, the equation becomes

(a) $D_x^2 u + D_y^2 u = 0$.

For the problem given it is subject to the conditions,

(1) $u = 0$, when $x = 0$
(2) $u = 0$ when $x = \pi$
(3) $u = 0$ when $y = \infty$
(4) $u = 1$ when $y = 0$.

Take $u = e^{ay}e^{axi}$, a solution of (a), and $u = e^{ay}e^{-axi}$, another solution of (a); add these two values of $u$ and divide the sum by two, which gives

(b) $e^{ay} \cos ax$.

From the solution given in (b), it is possible to arrive at the solution,

(c) $u = e^{ay} \sin ax$.

Out of these particular solutions we can build up a solution which will satisfy the conditions (1), (2),
(3), and (4). If $u$ is written equal to a sum of terms of the form $Ae^{-my} \sin mx$, where $m$ is a positive integer, a solution of (a) which satisfies (1), (2) and (3) is found. This solution is written in the form of a series

\[(d)\ u = A_1 e^{-y} \sin x + A_2 e^{-2y} \sin 2x + \ldots,\]

$A_1, A_2, A_3, \ldots$, being undetermined constants.

When $y = 0$, (d) reduces to

\[(e)\ u = A_1 \sin x + A_2 \sin 2x + A_3 \sin 3x + \ldots.\]

If unity is developed into a series of the form (d) the problem is solved. All that needs to be done is substitute the coefficients of that series for $A_1, A_2, A_3, \ldots$, in the equation (b).

A sine development in series is obtained by setting $f(x) = x$, It is

\[x = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \ldots,\]

which is a Fourier's series. By the process of integration and substitution, we arrive at a solution which is a form of Fourier's infinite series. It is

\[1 = 4/\pi ((\sin x)/1 + (\sin 3x)/3 + (\sin 5x)/5 + \ldots),\]

for all values of $x$ between 0 and $\pi$; hence the required solution is

for this satisfies the differential equation and all the given conditions.

The problem of finding the value of the potential function at any point of a long thin, rectangular conducting sheet, of breadth \( \pi \), through which an electric current is flowing, when the long edges are kept at potential zero, and one of the short edges at potential unity, is mathematically identical with the problem just solved.¹

The problem of the transverse vibrations of a stretched string fastened at the ends, initially distorted into some given curve and then allowed to swing will be solved.

Let the length of the string be \( L \) and take the position of equilibrium of the string as the axis of \( x \), and one of the ends as the origin, and suppose the string initially distorted into a curve whose equation is

\[ y = f(x). \]

The next thing is to find an expression for \( y \) which will be a solution of the equation and satisfy the con-

---

¹ Byerly, Fourier's Series, p. 5, 6.
ditions given.

\[ (1) \quad D_t^2 y = a^2 D_x^2 y \]

(a) \( y = 0 \) when \( x = 0 \)
(b) \( y = 0 \) when \( x = L \)
(c) \( y = f(x) \) when \( t = 0 \)
(d) \( D_t y = 0 \) when \( t = 0 \).

In this instance it is convenient to use the trigonometric form of solution. The equation is, if \( t = 0 \),

\[ (2) \quad y = A_1 \sin \frac{\pi x}{a} + A_2 \sin 2\pi x/a + \cdots \]

a solution of (1) which satisfies (a), (b), and (d).

If \( f(x) \) is developed into a series of form (2),
the coefficients of this series as values of \( A_1, A_2, A_3, \ldots \), substituted in,

\[ (3) \quad y = A_1 \sin \frac{\pi x}{a} \cos \frac{n\pi t}{a} + A_2 \sin 2\pi x/a \cos 2\pi n t/a + \cdots \]
gives a solution which satisfies all the given conditions. 1

The complete solution is

\[ (4) \quad y = 2/L \sum_{m=0}^{\infty} \sin \frac{m\pi x}{L} \cos \frac{m\pi x}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} \, dx \]

The second member of (4) is a periodic function of \( t \) having the period \( 2L/a \). The motion, then, is a true

vibration, a periodic motion. The period $2L/a$ is the time it takes a disturbance to travel twice the length of the string.

If the curve into which the string is distorted initially is

$$y = b \sin \frac{m\pi x}{L},$$

the solution is

$$y = b \sin \frac{m\pi x}{L} \cos \frac{m\pi at}{L}.$$  

No matter what value $t$ may have the curve is always of the form

$$y = A \sin \frac{m\pi x}{L}.$$  

For different values of $t$, there is a set of sine curves differing only in amplitude and not at all in the period of the curve.  

The General Electric Company uses Fourier's series in handling problems in electric circuits, providing the electric current or voltage is irregular in form but repeats at definite intervals. It is represented by a Fourier series as follows;

$$e = \sum_{n=0}^{\infty} a_n \cos n\omega t + \sum_{n=0}^{\infty} b_n \sin n\omega t.$$  

If the series is rectangular in form, the Fourier series becomes,

$$i = 4I/\pi (\sin \omega t + 1/3 \sin 3\omega t + 1/5 \sin 5\omega t + \cdots).$$

The problem of a stretched circular membrane such

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as a drumhead will be discussed. If the membrane is initially distorted into any given form which has circular symmetry about an axis through the center perpendicular to the plane of the boundary, and then allowed to vibrate. The equation which follows has to be solved

\[(1) \frac{D_t^2 z}{c^2} = \frac{D_r^2 z}{(1/r)} + \frac{D_r^2 z}{(1/r^2)} + \frac{D_0^2 z}{D_0^2} ,\]

subject to the conditions

(a) \( z = f(r) \) when \( t = 0 \)

(b) \( D_t z = 0 \) when \( t = 0 \)

(c) \( z = 0 \) when \( r = a \).

From the symmetry of the supposed initial distortion \( z \) must be independent of \( \theta \), therefore (1) reduces to

\[(2) \frac{D_t^2 z}{c^2} = \frac{D_r^2 z}{(1/r)} + \frac{D_r^2 z}{(1/r^2)} .\]

The problem is best solved by the use of a Bessel's function, where the series is used in the solution. The series solution is of the form,

\[ R = a_0 (1 - x^2/2^2 + x^4/2^4 2^2 + \ldots) \]

where \( a_0 \) may be taken as any value. \(^1\)

Take \( a_0 = 1 \), and then \( R = J_0(x) \), where

\[ J_0(x) = 1 - x^2/2^2 + x^4/2^4 2^2 + \ldots \]

---

is a solution of (2). \( J_0(x) \) is easily shown to be convergent for all values real or imaginary of \( x \), since the series is made up of the moduli of the terms of \( J_0(x) \), it is convergent for all values of \( r \).

\( J_0(x) \) is called a Bessel's Function of the zero-th order, or a Cylindrical Harmonic. It can be shown that \( J_0(x) = 0 \) has an infinite number of real positive roots, any one of which can be obtained to any required degree of approximation without serious difficulty.\(^1\)

The General Electric Company uses the special type of series called Bessel's Function, when dealing with problems involving circles such as the heating or cooling of a large shaft, the flow of alternating current in a circular wire, etc. This series is of the form

\[ J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^24^2} - \ldots \]

This series occurs so frequently that tables have been formed for some ranges of the variables \( x \).\(^2\)

The use of the infinite series called Bessel's function is quite common in the field of radio.

The problem which follows is one concerning the power radiated from the flat top of a sending antenna. The element of the area of the aerial hemisphere is,

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(1) \( ds = r_e^2 \sin u \, du \, ds \).

By the process of substitution in the differential equation

\[
(2) \, dp = c/4\pi \left( E_u H_s + E_s H_u + 2 \cos a E_u H_s \right) \, ds,
\]

and the replacing of the value \( z \) by its value in polar coordinates, we have

\[
(3) \, dp = c/4\pi (E_u H_s) \, ds.
\]

The integral form of (3) becomes if \( \bar{p} \) = the time average of radiated power, and the equation thus formed is integrated with respect to \( s \),

\[
(4) \, \bar{p} = \pi/2 - 1/2 \int_0^\pi \cos (2A \sin u \cos s) \, ds.
\]

By the use of Bessel's function of the zero-th order,

\[
(5) \, J_0(x) = 1 - x^2/2^2 + x^4/2^2 4^2 - \ldots,
\]

and substituting and integrating a result is found which may be expanded in a power series. This gives

\[
(6) \, \bar{p} = I^2/c \cdot S(-1)^n/2 \cdot k^n/n! \cdot F_n(B),
\]

where \( F_n(B) \) is a polynomial in \( B^0, B^2, B^4, \ldots \), where the coefficients of the several powers of \( B \) are taken from tabular form.

If we substitute the value of the coefficients multiplied by the corresponding powers of \( B \) and sum up, the time average of radiated power is found.
\[ \bar{P} = \frac{I^2}{c} (k^2 (B^4/60 - \frac{11B^6}{3780} + \frac{13B^8}{56700} - \frac{B^{10}}{9355} - \ldots) - k^2 (\frac{B^4}{1120} - \frac{B^6}{6480} + \frac{B^8}{83160} - \frac{B^{10}}{7739550} + \ldots) + k^6 (\frac{B^4}{45360} - \frac{B^6}{24960960} + \frac{7B^8}{634720} - \ldots)) \]

This equation gives the average power radiated in the aerial hemisphere from the flat top of the antenna regarded as a separate radiator with the distribution that it has under the fundamental assumptions of the problem. The current is to be measured in absolute electrostatic units, and the power in ergs per second.

In the equation above

\[ B = \frac{2\pi b}{L} \]
\[ k = \frac{2\pi a}{L} \]
\[ L = \text{the wavelength with the inductance} \]
\[ a = \text{length of vertical part of antenna in cm.} \]
\[ b = \text{length of horizontal flat top in cm.} \]

The General Electric Company uses infinite series of various other kinds and forms and some examples of these will be given in the following. In the calculation of Eddy currents certain integrals arise that can sometimes be evaluated in terms of the so called "integral exponentials".

\[ \text{Ei}(x) = \int_{-\infty}^{x} e^{-u/u} du = c + \ln x + x + \frac{x^2}{2} \cdot \frac{2}{2} + \ldots, \]

where \( c = \text{Euler's Constant} = 0.5772156 \ldots \).

The next series deals with the eddy current loss due to induced currents in the iron sheath of a three conductor cable. The applied field due to current in the three conductors is,

\[
E_a = -3/2 \left( (z_0 e^{j\theta})/r + (z_0^2 e^{-2j\theta})/2r^2 + (z_0^4 e^{4j\theta})/4r^4 + (z_0^5 e^{-5j\theta})/5r^5 \right).
\]

The induced electric field is

\[
E_{\text{ind}} = (2 \cdot 10^{-9} j\omega I)(3/2) \frac{z_0 e^{j\theta} r}{a^2} \frac{(1 + Aa/u)}{(1 - Aa/u)} + \frac{z_0^2 e^{-2j\theta} r^2}{a^4} \frac{(2 + Aa/u)}{(2 - Aa/u)}.
\]

The above series is rather difficult to calculate as nearly all of the quantities are complex.\(^1\)

In the theory of probability there are a great number of uses for series. We will solve the problem of the selection of balls from a box. A box contains \( n \) balls, marked 1, 2, 3, \ldots, \( n \). A person draws \( n \) balls in succession and none of the balls thus drawn is put back in the box. Each drawing is consecutively marked 1, 2, 3, \ldots, \( n \), on \( n \) cards. What is the probability that no ball marked \( a \) (\( a = 1, 2, 3, \ldots, n \)), appear

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simultaneously with a drawing card marked?

The probability that none of the balls appear in its numerical order is

\[ p = 1 - \frac{1}{1} + \frac{1}{1 \cdot 2} - \frac{1}{1 \cdot 2 \cdot 3} + \cdots (-1)^n \frac{1}{1 \cdot 2 \cdot \cdots n}. \]

When \( n = \infty \) the above series converges toward \( e^{-1} \) as its limit. Since the series is rapidly convergent, its approximate value is

\[ p = e^{-1} = 0.36788 \]

The probability that at least one ball appear in numerical order is

\[ q = 1 - p = 0.63213 \]

The discussion which follows will deal with the use of series in the solution of differential equations. If the differential equation is put in the form,

\[ (1) \frac{dy}{dx} = f(x, y). \]

Let \( x = x_0, y = y_0 \), and we obtain \( (dy/dx)_0, (d^2y/dx^2)_0, \) etc. The values are then substituted in the Taylor series,

\[ (2) y = y_0 + (dy/dx)_0(x - x_0) + \frac{1}{2!}(d^2y/dx^2)(x - x_0)^2 + \cdots, \]

which gives a solution of the given differential equation in the form of an infinite series. Another solution of (1), is to substitute the series that follows in (1),

\[ \cdots \]

and determine $a_1$, $a_2$, $a_3$, \ldots, by the method of undetermined coefficients.\(^1\)

Legendre's Equation is

\[(1) \ (1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0,\]

where $n$ is a constant. It is solved by assuming the infinite series

\[y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \cdots,\]

and endeavoring to determine the first exponent $m$ and the coefficients $a_i$ so as to satisfy (1). The general solution of (1) is the two series,

\[y = a_0(1 - n(n-1)x^2/2! + n(n-2)(n-1)(n-3)x^4/4! - \cdots) - a_1(x - (n-1)(n-2)x^3/3! + (n-1)(n-2)(n-3)(n-4)x^5/5! - \cdots).\]

By the ratio test each of these series converges in the interval $(-1, 1)$. This is the general solution of (1), since $a_0$ and $a_1$ are arbitrary.\(^2\)

Laplace's equation in two variables is

\[\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} = 0.\]

To find a particular solution which satisfies the given conditions, we place

\[V = XY\]

\(^1\) Woods, Advanced Calculus, p. 223.
\(^2\) Ibid, p. 268, 269.
where $X$ is a function of $x$ only, and $Y$ is a function of $y$ only. We have as the solution the summation of the series,

$$V = \sum_{k=0}^{\infty} \left( e^{kY} \cos kx + B_k \sin kx \right) + e^{-kY} \left( M_k \cos kx + N_k \sin kx \right).$$

In polar coordinates the solution of a two variable equation is of the form,

$$V = \sum_{k=0}^{\infty} \left( r^k (A_k \cos k\theta + B_k \sin k\theta) \right) + r^{-k} \left( M_k \cos k\theta + N_k \sin k\theta \right).$$

The equation in three variables is

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = 0.$$ 

This equation occurs in the flow of heat, and many other problems in physics. The general solution in cylindrical coordinates is,

$$V = \sum_{m=0}^{\infty} \left( e^{kz} (A_m \cos m\phi + B_m \sin m\phi) \right) + e^{-kz} \left( C_m \cos m\phi + D_m \sin m\phi \right) J_m(kr).$$

The solution for polar coordinates is obtained by the use of Legendre's equation and is,

$$V = \sum_{m=0}^{\infty} \left( A_m r^m + B_m/r^{m-1} \right) P_m(\cos \phi).$$

This series solution is used in the solving of problems in potential.

Conclusion

The problem of asymptotic series approximation is to find a close approximation of the function under consideration in terms of a finite sum. This sum is to represent the function over a finite interval and to be a close approximation to the function. The asymptotic series solution of problems is usually set up in the general form, and special solutions are derived.

The development of the asymptotic series was the result of the need for methods of close approximation to a finite function that could not be represented by a convergent series. Its development was slow in comparison with certain other branches of mathematics because it made use of the divergent series as a method of solution. The mathematicians before the nineteenth century thought divergent series could not be used in the mathematical solution of a problem. This idea of expressing an approximation only by a convergent series made it impossible for certain problems to be solved.

Euler in his development of the theory of asymptotic series made use of divergent series and showed that they gave a close approximation to the function being considered. The various writers in the nineteenth
century who contributed to the development of asymptotic series showed that in most instances the divergent series gave a better degree of approximation than the convergent series.

The applied field of mathematics, which has been greatly enlarged by inventions in comparatively recent years, was also a factor in the development of the asymptotic series. The fact that an asymptotic approximation may be derived is of importance in the applied fields of mathematics as the finite sum thus expressed is close enough for use in practical application.

In the future with our many new inventions and the enlargement of our field of applied mathematics, there are many reasons to believe that asymptotic series will play a greater part in the development of the mechanical, physical and radio world. This development will necessitate a wider and more varied use of asymptotic series.
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