A Brief Consideration of The Different Methods of Integration

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A Brief Consideration of the Different

Methods of Integration

being

A thesis presented to the Graduate Faculty of

Fort Hays Kansas State College in partial

fulfillment of the requirements for the

degree of

Master of Science

by

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November 12, 1937

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ACKNOWLEDGEMENTS

The author wishes to express his deepest appreciation to those who made this thesis possible, namely Professor E. E. Colyer who directed the work, Doctor W. G. Warnock for his valuable suggestions, and Doctor F. B. Streeter for materials ordered and technical advice given regarding organization and construction. The author further extends his gratitude to the Librarians of the Fort Hays Kansas State College for their assistance in locating sources of reference.

INTRODUCTION

Any student of mathematics is familiar with the importance of the process of integration. Integration is as fundamental to analysis as the basic principles of the number theory are to arithmetical calculation. Some of the common applications of integration are: finding the distance a falling body has travelled during a particular interval of time; to determine the equation of a curve, given different conditions (such as slope of the curve equal numerically to one-half the abscissa, or some similar problem); motion of a projectile; motion in a resisting medium; finding areas and volumes of revolution; length of a curve; areas of surfaces of revolution; work of expanding gases; and numerous other practical uses.

With these many useful applications in mind, the author chose the problem of studying the various methods of integration. It is his earnest desire to learn more about the theory of this interesting subject and to summarize briefly the more familiar definitions of the Riemann and Lebesgue integrals and then to consider less familiar modifications of these definitions. Because of the wealth of material on these subjects, it will be necessary to reduce the discussion of each integral to a minimum. Several important existence theorems will be proven for the Riemann and Lebesgue integrals; followed by a comparison of these two definitions. In discussing the modifications of the above definitions, the author will show the difference between the modification and previous definitions. Lastly, he will offer the opinions of several outstanding mathematicians of the present time regarding the possible trend of integration in the future.

In examining the abstracts of theses available in this library and that of the University of Kansas, the author found only one thesis previously written on integration, and that was a Doctor's thesis written on the "Stieltjes integral."

CHAPTER I

A Historical Development of Integration

A historical development of Integration would be an incoherent treatise if the author tried to compile data on that subject alone without treating the related subject of differentiation. Because of the close correlation between the Integral and Differential Calculus, the author will attempt to give a historical development of the two combined, stressing the integration where this will not affect the continuity of the discussion.

Zeno of Elea[1] (450 B. C.) was one of the first to introduce problems that led to a consideration of infinitesimal magnitudes. He argued that motion was impossible for this reason:

> Before a moving body can arrive at its destination it must have arrived at the middle of its path; before getting there it must have accomplished the half of that distance, and so on *ad infinitum*: in short, every body, in order to move from one place to another, must pass through an infinite number of spaces, which is impossible.[2]

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1. Smith, *History of Mathematics*, II, 667.
2. Ibid., 677 taken from George J. Allman, *Greek Geometry from Thales to Euclid*, 55.

Leucippus (c 440 B. C.) and Democritus (460-370 B. C.) are generally considered as the founders of the atomistic school,[3] which taught that magnitudes are composed of indivisible elements in finite numbers. It was this philosophy that led Aristotle[4] (340 B. C.) to write a book on indivisible lines in which he tried to show the mathematical and logical impossibility of this process. This book is also attributed to Theophrastus.

Antiphon (c 430 B. C.) is one of the earliest writers whose use of the method of exhaustion is fairly well known to us. This method of exhaustion was to inscribe a regular polygon in a circle and then, by bisecting the sides of the polygon and their subtended arcs, to double the number of sides until the perimeter of the polygon approached the circumference of the circle as its limit; thus exhausting the area between the polygon and the circle. This method of exhaustion was widely used by early Greek mathematicians. Later the polygon was circumscribed "to double" by continually doubling the number of sides until the perimeter became a circle. This was an early idea in the theory of limits which later was so important in the development of the calculus. Eudoxus of Cnidus[5] (408-355 B. C.) is probably the one who placed the

3. Ibid., 677 et. seq.
4. Allman, Greek Geometry from Thales to Euclid, 56.
5. Smith, op. cit. taken from Heath, Euclid, II, 120.

theory of exhaustion on a scientific basis. His method depends on the proposition[6] that "if from the greater of two magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there will at length remain a magnitude less than the least of the proposed magnitudes." In his definition he excludes the relation of a finite magnitude to a magnitude of the same kind which is either infinitely great or infinitely small. It is in this definition and the related axiom that Dr. Allman finds a basis for the scientific method of exhaustion and discerns the probable influence of Eudoxus.

It is to Archimedes himself (225 B. C.) that we owe the nearest approach to actual integration to be found among the ancient Greeks.[7] It would seem that Archimedes' mode of procedure[8] was, to start with mechanics (center of mass of surfaces and solids) and by his infinitesimal-mechanical method to discover new results for which later he deduced and published the rigorous proofs. His first noteworthy advance toward calculus was concerned with his proof that the area of a parabolic segment is 4/3 of a triangle with the same base and vertex, or two-thirds of the circumscribed parallelogram.[9]

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6. Ball, A short account of the history of Mathematics, 45.
7. Ibid., 679 taken from Heath, Works of Archimedes, cxlii and other valuable references.
8. Cajori, A History of Mathematics, 36.
9. Smith, op. cit., 679.

This was shown by continually inscribing in each segment between the parabola and the inscribed figure a triangle with the same base and the same height as the segment. If A is the area of the original inscribed triangle, the process adopted by him leads to the summation of the series

$$A + \tfrac{1}{4}A + (\tfrac{1}{4})^2A + (\tfrac{1}{4})^3A + ---,$$

or to finding the value of

$$A\left[1 + \tfrac{1}{4} + (\tfrac{1}{4})^2 + (\tfrac{1}{4})^3 + ---\right],$$

so that he really finds the area by integration and recognizes, but does not assert, that $(\frac{1}{4})^n \to 0$ as $n \to \infty$, this being the earliest example that has come down to us of the summation of an infinite series.

The only traces we have of an approach to calculus in the Middle Ages are those relating to mensuration and to graphs.[10] The idea of breaking up a plane surface into infinitesimal rectangles was probably present in the minds of many of the mathematicians of the time, but it was never elaborated into a theory that seemed worth considering. Jehudah Barzilai,[11] a Jewish writer living in the thirteenth century, asserts that

> "It has been said that there is no form in the world except the rectangle, for every triangle or rectangle

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10. Ibid., 684.
11. Ibid., 684, 685 taken from Sefer Jezira, Commentary by Judah ben Barzilai, 255.

is composed of rectangles too small to be perceived by the senses."

Oresme[12] (c 1360) took the next important step in the preparation for the calculus of the Middle Ages. His method of latitudes and longitudes gave rise to what we would now call a distribution curve or graph. This step is fundamental to the modern method of finding the area included between a curve and certain straight lines.

Even as late as the middle or end of the sixteenth century no marked progress in calculus had been made from the time of Archimedes.[13] Statistics (of solids) and hydrostatics remained in much the state in which he had left them, while dynamics as a science, did not exist. As is usual in such cases, it is impossible to determine with certainty to whom credit belongs, in modern times, for first making any noteworthy move in calculus, but it is safe to say that Simon Stevinus[14] (1586) is entitled to serious consideration. His contribution is seen particularly in his treatment of the subject of the center of gravity of various geometric figures, anticipating, as it did, the work of several later writers.

Following the time of Stevinus the brightest and most

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12. Ibid., 319.
13. Ball, op. cit., 244.
14. Smith, op. cit., 685.

brilliant mathematicians bent the force of their genius in a direction which finally led to the discovery of the infinitesimal calculus by Newton and Leibniz.[15] Kepler, Cavalieri, Roberval, Fermat, Descartes, Wallis and others had each contributed to the new Cartesian geometry. So great was the advance made, and so near was their approach toward the invention of infinitesimal analysis, that both Lagrange and Laplace pronounced Fermat to be the first inventor of it. The differential calculus, therefore, was not so much an individual discovery as the grand result of a succession of discoveries by different minds.

What is considered by us as the process of differentiating was known to quite an extent by Isaac Barrow[16] (1663), a teacher of Isaac Newton. Barrow gave a method of tangents in which, in the annexed figure, Fig. I, Q approaches P as in our present theory, the result being an indefinitely small arc.

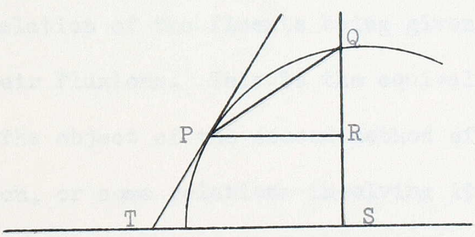

Fig. I.

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15. Cajori, op. cit., 191 et. seq.
16. Smith, op. cit., 690, 691.

It is quite probable that Barrow advised Newton of this figure as early as 1664.[17] Pascal had already published a figure of somewhat the same shape. The triangles given by both Barrow and Pascal were apparently known to Leibnitz and helped him in developing his own theory.

In 1665 and 1666 Isaac Newton[18] conceived the method of fluxions and applied them to the quadrature of curves. In his youth Newton studied Descartes' Geometry before he read Euclid. Thus, Descartes laid the foundation for Newton to build the calculus. Newton assumed that all geometrical magnitudes might be conceived as generated by continuous motion; thus a line may be considered as generated by the motion of a point, a surface by that of a line, and so on.[19] The quantity so generated was defined by him as the fluent or flowing quantity. The velocity of the moving body was defined as the fluxion of the fluent. In accordance with Newton's treatment of the subject there are two kinds of problems. The object of the first is, the relation of the fluents being given, to find the relation of their fluxions. This is the equivalent to differentiation. The object of the second method of fluxions is, from the fluxion, or some relations involving it, to determine the fluent. No account of Newton's method was published until 1693, though its general outline was known by his

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17. Ibid., taken from Child, Leibniz Manuscripts, 11.
18. Cajori, op. cit., 192.
19. Ball, op. cit., 344 et. seq.

friends and pupils before that time.

Lao G. Simons[20] presents an interesting discussion of the adoption of the method of fluxions in American schools. His study shows the almost complete dominance of the great Newton himself in American schools as far as the subject of fluxions is concerned. By the end of the first quarter of the nineteenth century, the catalogue of at least one college, Yale, shows that fluxions had been accorded a place among electives for the student body.

Leibniz[21] observed, in the study of Cartesian geometry, the connection existing between the direct and inverse problems of tangents. In 1673, while working upon the problems of tangents and quadratures, he invented a notation which was original and at the same time was generally more usable than that of Newton,--the "differential notation." He proposed to represent the sum of Cavalieri's indivisibles by the symbol $\int$, the old form of s, the initial of summa, using this with Cavalieri's omn. (omnia), to represent the inverse operation by d. By 1675 he had settled this notation, writing $\int y\,dy = \frac{1}{2}y^2$ as it is written at present. He published this method in 1684 and 1686 in Acta Eruditorium, a Berlin Journal, speaking of the integral calculus as the calculus summatorius. In

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20. Simons, <u>The adoption of the method of Fluxions in American schools</u>, 207 <u>et</u>. <u>seq</u>.
21. Cajori, <u>op</u>. <u>cit</u>., 207.

1696 he adopted the term calculus integralis, which name was decided upon with the help of Johann Bernoulli.[22] Leibniz' method of differences eventually supplanted, both in concepts and symbols, the fluxions of Newton.[23]

The early distinction between the systems of Newton and Leibniz lies in this, that Newton, holding to the conception of velocity or fluxion, used the infinitely small increment as a means of determining it, while with Leibniz the relation of the infinitely small increments is itself the object of determination.[24] The difference between the two rests mainly upon a difference in the mode of generating quantities.

The dispute between the friends of Newton and those of Leibniz as to priority of discovery was bitter and profitless. Even after the death of Leibniz in 1716 the controversy was bitterly debated for many years later. During the eighteenth century the prevalent opinion was against Leibniz but today the majority of writers are inclined to think that the inventions of Newton and Leibniz were independent.[25] An unfortunate result of this controversy was that until about 1820 the British mathematicians were ignorant of the brilliant mathematical discoveries on the continent. In 1813 the "Analytical Society" founded by Peacock, Herschel, and Babbage eliminated the flux-

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22. Smith, op. cit., 696.
23. Simons, op. cit., 207.
24. Cajori, op. cit., 197.
25. Ball, op. cit., 361.

ional notation of the calculus and opened to English students the vast storehouses of the continental discoveries.

In the seventeenth century a native calculus, yenri[26] (circle principle), was developed in Japan. This native calculus thought to have been invented by the great Seki Kowa was an application of series to the ancient method of exhaustion.

Outstanding mathematicians of the period from 1730 to 1820 were Euler, Lagrange, Laplace and Legendre.[27] Briefly Euler extended, summed up and completed the work of his predecessors; while Lagrange developed the infinitesimal calculus and theoretical mechanics, and presented them in forms similar to those in which we now have them. At the same time Laplace made some additions to the infinitesimal calculus and applied that calculus to the theory of universal gravitation; he also created a calculus of probabilities. Legendre published three works of elliptic integrals.

Men of note in the field of calculus for the period following 1820 might include Cauchy, Abel, Riemann, Weierstrass, DeMorgan and many others. Due to the increasing number of mathematical contributors in the latter part of the nineteenth century and early part of the twentieth century, the author will not attempt to trace the calculus further historically. Any further historical material needed in the development of

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26. Smith and Mikami, A History of Japanese Mathematics, 143 et. seq.
27. Ball, op. cit., 392.

this thesis will be added with the corresponding modification of the definition of integration.

CHAPTER II

Integration Defined and Explained

> The differential and integral calculus have to do with three fundamental notions[28] associated with functions, to which are due most of the applications of the function theory in geometry, mechanics, and physics, as well as other branches of science. These three conceptions are called the derivative, the antiderivative or indefinite integral, and the definite integral.

Two main types of problems of the differential calculus are construction of tangents to curves, and determination of the rate of change of a quantity. The fundamental definition[29] of the differential calculus is:

> The derivative of a function is the limit of the ratio of the increment of the function to the increment of the independent variable, when the latter increment varies and approaches the limit zero.

When the limit of this ratio exists, the function is said to possess a derivative. The above definition may be given in a more compact form symbolically as follows: Given

II - 1 $$y = f(x)$$

and consider x to have a fixed value. Let x take on an in-

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28. Young, Monographs on topics of modern Mathematics relevant to the elementary field, 285, article by Gilbert A. Bliss.

29. Granville, Elements of the Differential and Integral Calculus, 21, 22.

crement Δx; then the function y takes on an increment Δy, the new value of the function being

II - 2 $$y + \Delta y = f(x + \Delta x).$$

To find the increment of the function, subtract II - 1 from II - 2 giving

II - 3 $$\Delta y = f(x + \Delta x) - f(x)$$

Dividing by the increment of the variable, Δx, we get

II - 4 $$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x).}{\Delta x}$$

The limit of this ratio when Δx approaches the limit zero is, from our definition, the derivative which is denoted by the symbol dy/dx. Therefore

II - 5 $$\frac{dy}{dx} = \lim_{x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

defines the derivative of y [or f(x)] with respect to x if the limit exists.

From II - 4 we also get

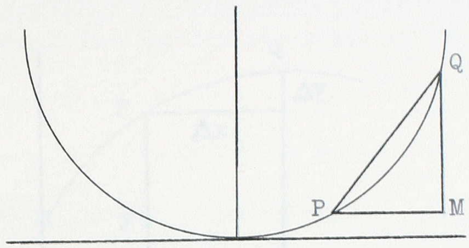
$$\frac{dy}{dx} = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$$

The process of finding the derivative of a function is called differentiation. It should be carefully noted that the derivative is the limit of the ratio, not the ratio of the limits. The latter ratio would assume the form 0/0, which is indeterminate. Since Δy and Δx are always finite and have definite values, the expression $\frac{\Delta y}{\Delta x}$ is really a fraction.

The symbol dy/dx is to be regarded not as a fraction but as the limiting value of a fraction.

Derivative is best brought out by considering the construction of the tangent to a curve.[30] Constructing the

Fig. II



parabola $y = x^2$, Fig. II, we wish to show the construction of a tangent to this curve at the point P. Let Q be any point on the parabola distinct from P. Join P to Q by a straight line, secant. Now let Q come closer and closer to P without reaching it. As Q approaches P the secant will rotate about P and will tend to coincide with a line through P which touches the parabola at P without cutting across the parabola. This line is tangent to the parabola at P. Q cannot come in coincidence with P, otherwise there could be no straight line.

By the slope of a line we mean the trigonometric tangent of the inclination of the line. The slope of the secant PQ has for a limit, as Q approaches P, the slope of the tangent

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30. Ritt, Differential and Integral Calculus, article in Encyclopedia Britannica Vol. IV, 555, 556.

at P. If PM and QM are perpendicular, Fig. II, the slope of PQ is $\frac{QM}{PM}$.

This brings us to a further conception of the derivative. Let $y = f(x)$ be any continuous function of x which is graphed in Fig. III. We choose any value for x and keep it fixed. To the fixed

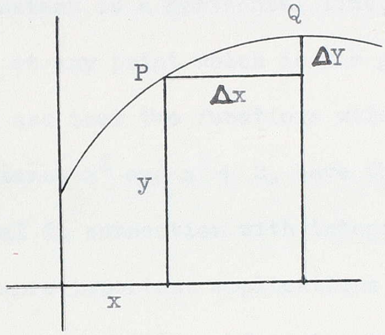

Fig. III

value of x corresponds a fixed value of y. This gives a fixed point P on the graph of coordinates (x,y). Now take any point Q of coordinates $(x + \Delta x, y + \Delta y)$. As Q approaches P, the ratio $\frac{\Delta y}{\Delta x}$ will approach a definite limit, the slope of the tangent to the graph at P. The limit of $\frac{\Delta y}{\Delta x}$ is called the derivative of f(x) for the value of x. Thus the derivative is the slope of the tangent to the graph. The derivative of the continuous function $y = f(x)$ is represented, for any value of x, by dy/dx.

Various treatises on calculus show how to differentiate all types of expressions by using formulas. The general formula for differentiating a problem of type $y = x^2$ is

Derivative of $x^n = nx^{n-1}$, , for all values of n.

If y is a constant, then, for every Δx, $y + \Delta y = y$, so that $\Delta y = 0$. Hence $\frac{\Delta y}{\Delta x}$ is always zero, so that according to the definition of the limit of a constant quantity, dy/dx is also zero. This is brought out geometrically by the fact that the graph of a constant is a horizontal line, so that the tangent to the graph at any point which is the graph itself, has a zero slope. We see that two functions which differ by a constant, for instance x^2 and $x^2 + 2$, have the same derivative. This is fundamental in connection with integration.

One of the more important applications of differentiation is the solution of problems of mixima and minima. Consider the continuous function $y = f(x)$ which has a derivative for every x, Fig. IV. At a point such as A, at which the continuous function is a maximum, or at a point such as B, where

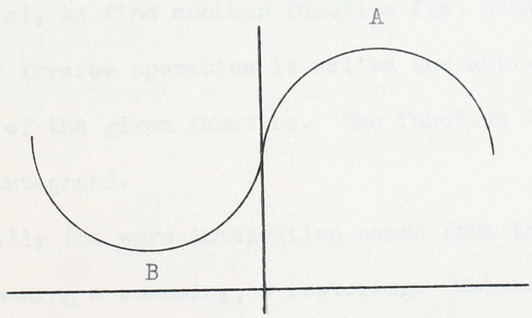

Fig. IV

the function is a minimum, the tangent is horizontal, the slope of the tangent is zero. This fact could be proved as follows:

Let the abscissa of A, be a, and suppose that dy/dx is not zero for x = a. Let us suppose that dy/dx is positive to a. When Δx is small, $\frac{\Delta y}{\Delta x}$ is very nearly equal to dy/dx. If Δx is small, $\frac{\Delta y}{\Delta x}$ is positive, like dy/dx. If Δx is small and positive, Δy must be positive, for if Δy were zero or negative, $\frac{\Delta y}{\Delta x}$ would be zero or negative. Since the function is continuous and has a continuous derivative, dy/dx can't be infinity. Therefore a point on the graph to the right of A must have a greater ordinate than A, so that y cannot be a maximum for x = a. This absurdity shows that dy/dx = 0 for x = a.

We next undertake a study of the second fundamental notion of the calculus, that of the anti-derivative or indefinite integral. In differential calculus we were chiefly concerned in finding the derivatives of given functions. We shall now consider the inverse operation;[31] that is, having given a function $\phi(x)$, to find another function f(x) such that $D_x f(x) = \phi x$. This inverse operation is called the anti-derivative or integral of the given function. The function integrated is called the integrand.

Literally the word integration comes from the Latin "integratio" meaning a renewing, a restoring. Webster's dictionary defines integration as

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31. Townsend and Goodenough, Essentials of Calculus, 101.

the inverse of the differentiation or derivation; also the doctrine of the limit of a sum of infinitesimals of which the number increases while the magnitude of each decreases; both without limit, but according to some law.

J. I. Hutchinson[32] defines integral calculus as

a branch of infinitesimal calculus treating of the methods of deducing relations between finite values of variables from given relations between contemporaneous infinitesimal elements of those variables. Its object is to discover the primitive function from which a given differential coefficient has been derived. This primitive function is called the integral of the proposed differential coefficient, and is obtained by the application of the different principles established in finding differential coefficients and by various transformations. To illustrate: with the integral calculus one may discover the relations connecting finite values of variables, as x and y, from the relation connecting their differentials, as dx and dy. Thus, integral calculus is the doctrine of the limit of the sum of infinitesimals of which the number increases while the magnitude decreases, both without limit, yet according to some law.....The sign of integration is "$\int$" which is a form derived from the old or long "s." It is the initial of the word "sum," and came into use owing to the conception that integration is the process of summing an infinite series of infinitesimals.....With the integral calculus a mathematician endeavors to transform the given expressions into others which are differentials of known functions and thus deduce formulas which may be applied to all similar forms.

It is the universal custom to denote integration by placing the symbol $\int$ before the differential. Since

$$d(x^3) = 3x^2\ dx$$

we write $\int 3x^2\ dx = x^3 + C$ where C is an arbitrary constant. The differential dx indicates that x is the independent variable.

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32. Hutchinson, Integral calculus, article in Encyclopedia Americana Vol. XV, 202.

Our problem[33] now becomes: "Having given the differential of a function, to find the function itself." Since integration and differentiation are inverse operations it follows that

$$\text{since } d(x^3) = 3x^2\,dx, \text{ we have } \int 3x^2\,dx = x^3;$$

$$\text{since } d(x^3 + 2) = 3x^2\,dx, \text{ we have } \int 3x^2\,dx = x^3 + 2;$$

$$\text{since } d(x^3 - 7) = 3x^2\,dx, \text{ we have } \int 3x^2\,dx = x^3 - 7.$$

In fact, since $d(x^3 + C) = 3x^2\,dx$ where C is any arbitrary constant, we have

$$\int 3x^2\,dx = x^3 + C$$

where C is a constant of integration independent of the variable of integration. Since we can give C as many values as we please, it follows that if a given differential expression has one integral, it has infinitely many differing only by constants. Hence

$$\int f'(x)\,dx = f(x) + C;$$

and since C is unknown and indefinite, the expression

$$f(x) + C$$

is called the indefinite integral of $f'(x)\,dx$.

If $\phi(x)$ is a function the derivative of which is $f(x)$, then $\phi(x) + C$, where C is any constant whatever, is likewise a function the derivative of which is $f(x)$. Hence the theorem:

If two functions differ by a constant, they have the same derivative.

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33. Granville, *op. cit.*, 189 *et. seq.*

The word indefinite refers to the fact that an arbitrary constant is involved in the integral.

Our discussion next brings up the definite integral. Before beginning the explanation of the definite integral it is necessary to prove

> that the differential of the area bounded by any curve, the x-axis, and the two ordinates is equal to the product of the ordinate terminating the area and the differential of the corresponding abscissa.[34]

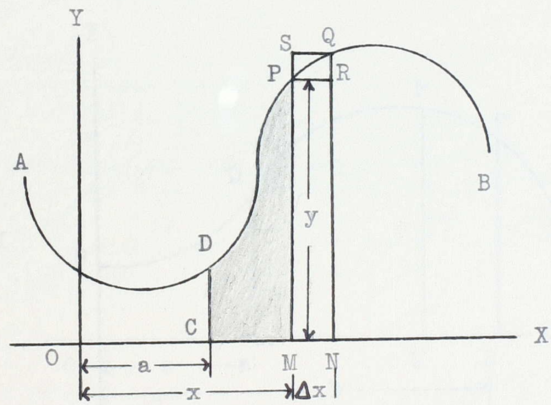
Consider the continuous function $\phi(x)$ and let

$$y = \phi(x)$$

be the equation of the curve AB, Fig. V.

Let CD be a fixed and MP a variable ordinate, and let u be the measure of the area CMPD. When x takes on

Fig. V



a sufficiently small increment Δx, u takes on an increment Δu(= area MNQP). Completing the rectangles MNRP and MNQS, we see that area MNRP < area MNQP < area MNQS

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34. Ibid., 237-239.

or, $MP. \Delta x < \Delta u < NQ. \Delta x$;
and dividing by Δx,

$$MP < \frac{\Delta u}{\Delta x} < NQ$$

If MP happens to be $>$ NQ, we simply reverse the inequality signs.

Now let $\Delta x \to 0$ as a limit; then since MP remains fixed and NQ approaches MP as a limit (since y is a continuous function of x), we get

$$du/dx = y \ (= MP),$$

or using differentials,

$$du = ydx.$$

which proves the theorem.

Now if $y = \phi(x)$
then $du = ydx$, or

II-6 $du = \phi(x)\, dx$,

where du is the differential of the area between the curve, the X-axis, and any two ordinates. Integrating II-6 we get

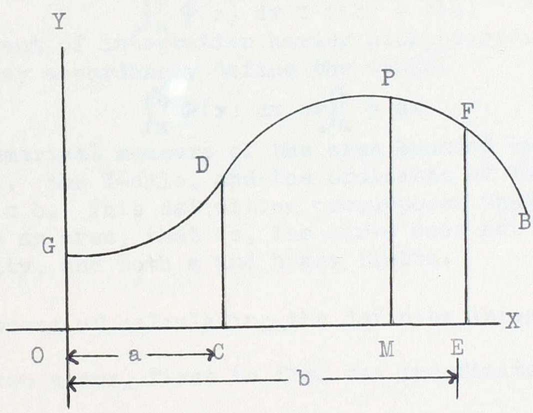
$$u = \int \phi(x)\, dx$$

Since $\int \phi(x)\, dx$ exists as an area, we denote it by $f(x) + C$

II-7 Therefore $u = f(x) + C$

We may determine C if we know the value of u for some value of x. If we agree to reckon the area from the axis of y, that is, Fig. VI, when

Fig. VI



II - 8 $x = a$, $u =$ area OCDG
and when $x = b$, $u =$ area OEFG, etc., it follows that
II - 9 if $x = 0$, then $u = 0$.
Substituting II-9 in II-7 we get

$$0 = f(0) + C, \text{ or } C = -f(0)$$

Hence from II-7 we obtain

II - 10 $u = f(x) - f(0)$

giving the area from the axis of y to any ordinate (as MP).

To find the area between the ordinates CD and EF, substitute the values II-8 in II-10, giving

II - 11 area OCDG $= f(a) - f(0)$.

II - 12 area OEFG $= f(b) - f(0)$.

Subtracting II - 11 from II - 12,

II - 13 area CEFD $= f(b) - f(a)$

Theorem:

The difference of the values of $\int y\,dx$ for $x = a$ and $x = b$ gives the area bounded by the curve whose ordinate is y, the X-axis and the ordinates corresponding to $x = a$ and $x = b$.

This difference is represented by the symbol

II - 14 $$\int_a^b y\,dx, \text{ or } \int_a^b \phi(x)\,dx,$$

and is read "the integral from a to b of y dx." The operation is called integration between limits, a being the lower and b the upper limit.

Since II - 14 always has a definite value, it is called a definite integral. For if

$$\int \phi(x)\,dx = f(x) + C,$$

then

$$\int_a^b \phi(x)\,dx = \Big[f(x) + C\Big]_a^b$$
$$= \Big[f(b) + C\Big] - \Big[f(a) + C\Big],$$
$$\int_a^b \phi(x)\,dx = f(b) - f(a)$$

the constant of integration having disappeared.

We may accordingly define the symbol

$$\int_a^b \phi(x)\,dx \text{ or } \int_a^b y\,dx$$

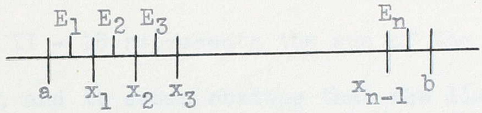
as the numerical measure of the area bounded by the curve $y = \phi(x)$. the X-axis, and the ordinates of the curve at $x = a$, $x = b$. This definition presupposes that these limits bound an area, that is, the curve does not rise or fall to infinity, and both a and b are finite.

The process of calculating the definite integral may be summed into two steps, first to find the indefinite integral of the given differential expression, and secondly to substitute in this indefinite integral first the upper limit and then the lower limit for the variable, and subtract the last result from the first. The constant of integration need not be introduced, because it always disappears in subtracting.

Woods[35] gives a graphical discussion of the method of obtaining the definite integral and then proves the existence of the limit under certain conditions. The graphical discussion only will be taken up in this treatise.

In the interval $a \leqq x \leqq b$, Fig. VII, assume n points $x_0 = a, x_1, x_2, x_3, \ldots\ldots, x_{n-1}$, where $x_{i+1} > x_i$, thus dividing (a,b) into

Fig. VII



n smaller intervals. In each interval take $x = E_i$, where $x_{i-1} \leqq E_i \leqq x_i$, and form the sum

$$\text{II - 15} \qquad \sum_{i=0}^{i=n-1} f(E_{i+1})\,(x_{i+1} - x_i) = f(E_1)\,(x_1 - a) + f(E_2)\,(x_2 - x_1) + \ldots + f(E_n)\,(b - x_{n-1}).$$

Now let n increase indefinitely while each of the n intervals $(x_{i+1} - x_i)$ approaches zero. If II - 15 approaches a limit independent of the choice of x_1 or E_i, that limit is called the definite integral of f(x) between a and b and is denoted by

$$\int_a^b f(x)\,dx$$

It may be made graphically plausible that the limit exists if f(x) is continuous and a and b are finite. If f(x) is expressed as a graph, we have a figure like Fig. VIII.

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35. Woods, Advanced Calculus, 134-137.

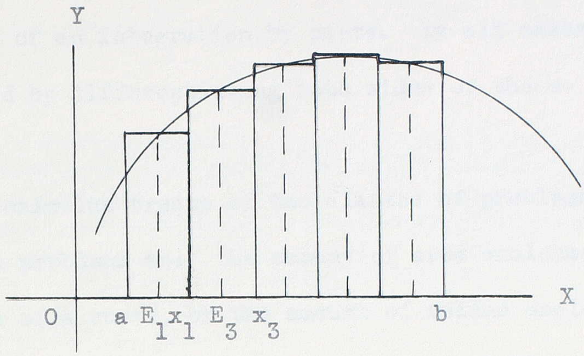

Fig. VIII

The sum II - 15 represents the sum of the rectangles of the figure, and it seems obvious that the limit of the sum is the area bounded by the curve, the X-axis, and the ordinates x = a, and x = b.

Also, if f(x) has a finite number of finite discontinuities, but a and b are finite, as in Fig. IX, the area and the integral seem to exist.

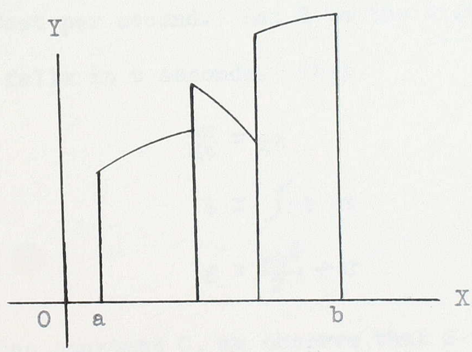

Fig. IX

As in the differential calculus different authors give formulas for integration. In some cases they represent an integral which has already been evaluated, and in other cases they are the result of an integration by parts. In all cases they can be varified by differentiating both sides of the equation.

The integral calculus treats of two classes of problems.[36] It first deals with problems as: the amount of area enclosed by a curve, the length of a curve, or the amount of volume enclosed by a surface; and secondly the determination of a variable quantity when the law of its change is known.

An example of the first class of problems has already been worked out in this treatise in connection with Fig. VI. A good example of the second type would be the problem of finding a formula for the distance through which a body, under the influence of gravity, falls, in any period of time. Let the body, initially at rest, be allowed to fall. If g is the acceleration of gravity, (32 ft./ sec.^{2}) the body will, in t seconds, acquire a speed of gt feet per second. Let S be the distance through which the body falls in t seconds. Then

$$\frac{ds}{dt} = gt$$

$$S = \int gt\, dt$$

$$S = \frac{gt^2}{2} + C$$

To find the constant C, we observe that S = 0, when t = 0.

Then $$0 = \frac{g0^2}{2} + C \text{ so that } C = 0.$$

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36. Ritt, op. cit., article in Encyclopedia Britannica Vol. IV, 557,558.

Hence, $$S = \frac{gt^2}{2} \quad \text{for every } t.$$

Ball[37] gives a very excellent summary of the preceding discussion:

> Wherever a quantity changes according to some continuous law--and most things in nature do so change--the differential calculus enables us to measure its rate of increase or decrease; and, from its rate of increase or decrease, the integral calculus enables us to find the original quantity. Formerly every separate function of x such as $(1 + x)^n$, $\log (1 + x)$, $\sin x$, $\tan^{-1}x$, etc. could be expanded in ascending powers of x only by means of such special procedure as was suitable for that particular problem; but by the aid of the calculus, the expansion of any function of x in ascending powers of x is in general reducible to one rule which covers all cases alike. So again, the theory of maxima and minima, the determination of the lengths of curves and the areas enclosed by them, the determination of surfaces, of volumes, and of centers of mass, and many other problems, are each reducible to a single rule. The theories of differential equations, or the calculus of variations, of finite differences, etc., are the developments of the ideas of the calculus.

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37. Ball, op. cit., 265.

CHAPTER III

The Riemann Integral

As stated in Chapter II, the integral calculus arose from attempts to find the lengths of curves, the area of curved or convex surfaces and the volume of irregular solids. The elementary properties of an integral show the integral first considered as the inverse process of differentiation and later as the limit of the sum of an indefinitely large number of small elements. The first notion[38] was used to evaluate the integral, the latter was best used in setting up an integral from given data.

A rigorous treatment of the integration notion dates from the time of Cauchy and Riemann. The definition of Cauchy covered the case for continuous functions. Riemann extended the Cauchy definition to bounded functions, and he also set up the condition for the existence of such an integral. Later the definitions of Lebesgue, Stieltjes, Young and others extended the Cauchy-Riemann definition to make it applicable to unbounded functions and to integration over unbounded

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38. Townsend, Functions of Real Variables, 198.

intervals. The Cauchy-Riemann definition is the one commonly employed in elementary analysis and in the applications to the physical sciences. The Riemann integral will probably continue to be the basis upon which practical applications of the integral calculus rest.[39]

The author will set up the definition of the Riemann integral and prove some existence theorems for this integral. Let $f(x)$ be a bounded function, defined for the interval (a,b).[40] Suppose this interval to be divided by the insertion between $a(\equiv x_0)$ and $b(\equiv x_n)$ of the intermediate points

$$o \equiv x_1,\ x_2,\ \text{--------},\ x_{n-1}.$$

Form the sum

$$S\ (o) \equiv (x_1 - x_0)\ f(E_1) + (x_2 - x_1)\ f(E_2) + \ldots + (x_n - x_{n-1})\ f(E_n)$$

$$\equiv \sum_{k=1}^{n} f(E_k)\ \Delta_k x,$$

where E_k is any point in the interval $(x_k - x_{k-1}) \equiv \Delta_k x$. The Riemann integral may now be defined as the limit

$$\text{III - 1} \qquad \underset{\Delta \to 0}{L} \sum_{k=1}^{n} f(E_k)\ \Delta_k x,$$

providing the value of this limit is independent of the manner of inserting the intermediate points o, and Δ is the largest of the $\Delta_k x$'s, frequently called the norm of the given set of

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39. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, I, 460.
40. Townsend, *op. cit.*, 198, 199.

intervals. As Δ approaches zero, n becomes infinite. Symbolically the Riemann integral is represented by $\int_a^b f(x)\,dx$. In passing to the limit, we note that the number of points inserted in each subinterval increases indefinitely as the norm Δ approaches zero.

This definition is equivalent to saying that for every arbitrarily small positive number e there exists a positive number d, depending on e, such that for every choice of E_k in the interval $\Delta_k x$ and for every subdivision whose norm satisfies the condition $\Delta < d$, we have

$$\left| \int_a^b f(x)\,dx - \sum f(E_k)\,\Delta_k x \right| < e$$

The function f(x) as defined is bounded in the given interval and the limits of integration are both finite. Integrals arising under these conditions are called finite or proper integrals.

The investigation of the necessary and sufficient conditions that the bounded function f(x) may have an R-integral in (a,b) is simplified by the introduction of the upper and lower R-integrals of the function f(x) in the interval (a,b).

Darboux first introduced the upper and lower integrals and rigorously defined them.[41] Denote by L_k and l_k respectively the least upper bound and the greatest lower bound of f(x) in the interval $\Delta_k x \equiv (x_k - x_{k-1})$. Form the two sums

41. Ibid., 200 footnote et. seq.

$$\overline{S}(o) = \sum L_k \Delta_k x, \quad \underline{S}(o) = \sum l_k \Delta_k x.$$

The values of $\overline{S}(o)$, $\underline{S}(o)$ depend upon the manner in which the given interval (a,b) is subdivided by the insertion of the intermediate points o. However for every method of subdivision we have

III - 2 $\qquad \overline{S}(o) \geqq l(b - a), \quad \underline{S}(o) \leqq L(b - a),$

where L, l are respectively the least upper bound and the greatest lower bound of $f(x)$ in (a,b). The aggregate of values which $\overline{S}(o)$ may have by all possible methods of subdivision of (a,b) by the insertion of intermediate points has a greatest lower bound which we call the upper integral of $f(x)$ in the interval (a,b). Likewise the sum $\underline{S}(o)$ has for all possible subdivisions of (a,b) a least upper bound which is called the lower integral of $f(x)$ in (a,b). We shall denote these integrals by

$$\overline{\int_a^b} f(x)\, dx, \quad \underline{\int_a^b} f(x)\, dx, \text{ respectively}$$

Theorem: If $f(x)$ is bounded in the interval (a,b), then

$$\underset{\Delta \to 0}{L} \overline{S}(o) = \overline{\int_a^b} f(x)\, dx, \quad \underset{\Delta \to 0}{L} \underline{S}(o) = \underline{\int_a^b} f(x)\, dx.$$

By different ways of inserting the intermediate points o in the given interval (a,b) we have different sets of subintervals Δ_k and may obtain different sums $\overline{S}(o)$. As mentioned before, the aggregate of values $S(o)$ has the upper integral $\overline{\int_a^b} f(x)\, dx$ for its greatest lower bound. We show that

III - 3 $$\underset{\Delta \to 0}{L}\ \overline{S}(o) = \overline{\int_a^b} f(x)\,dx,$$

Δ being the norm of the Δ_k's for any method of subdivision.

Since $\overline{\int_a^b} f(x)\,dx$ is the greatest lower bound of $\overline{S}(o)$ for all possible method of subdivision of the given interval, it follows that we can find a particular method of subdivision for which the sum

$$\overline{S}(o) = \sum_{k=1}^{m} L_k\, d_k$$

satisfies the relation

III - 4 $$\overline{S}(o_0) < \overline{\int_a^b} f(x)\,dx + e/2$$

where e is an arbitrarily small positive number and d_k is the length of the subinterval.

Taking any other method of subdivision, we can so select the points o, and hence $\Delta_k \equiv x_k - x_{k-1}$, that for this new method of subdivision the norm Δ shall be small enough to satisfy the inequality

III - 5 $$(m + 1)\, L \cdot \Delta < e/2,$$

where as before L is the least upper bound of f(x) in (a,b). Some of the resulting intervals Δ_k may lie wholly within a d_k interval while others may contain portions of two or more of the d_k intervals. Those Δ_k intervals falling in the first class contain no end-points of a d_k interval, while those of the second class each contain at least one end-point of a d_k interval. There can not be more than $(m + 1)$ of the Δ_k intervals of the second class and from the above inequality these contribute less than $e/2$ to the sum $S(o_0)$. For each of the Δ_k intervals of the first class we obtain a product $L_k \Delta_k$, which is less than the corresponding product that enters into the sum $\overline{S}(o_0)$. Consequently from III - 4 we have for all methods of subdivision of (a,b) and for every arbitrarily small value e

$$\overline{S}(o) \leq \overline{\int_a^b} f(x)\, dx$$

But since $\overline{\int_a^b} f(x)\, dx$ is the greatest lower bound of $\overline{S}(o)$, this relation is equivalent to the saying that

$$\underset{\Delta \to 0}{L}\ \overline{S}(o) = \overline{\int_a^b} f(x)\, dx.$$

Since this limiting value is independent of the method by which the given interval is subdivided by the insertion of intermediate points, the theorem follows. A similar proof would establish the existence of the lower integral.

We shall next consider the conditions which must be imposed upon the function $f(x)$ in order that the limit

$$\underset{\Delta \to 0}{L} \sum_{k=1}^{n} f(E_k)\ \Delta_k x$$

may exist.[42]

Theorem (a). Given a function of $f(x)$ which is bounded in the interval (a,b). A necessary and sufficient condition that the integral $\int_a^b f(x)\, dx$ exists is that

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42. Ibid., 209, 210.

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

This condition is sufficient because on consideration of the inequality,

III - 6 $$\underline{S}(o) \leqq S(o) \leqq \overline{S}(o),$$

where $\underline{S}(o)$, $S(o)$, $\overline{S}(o)$ have the values given them earlier in the chapter. Since by any subdivision we have

$$\lim_{\Delta \to 0} \underline{S}(o) = \underline{\int_a^b} f(x)\,dx, \qquad \lim_{\Delta \to 0} \overline{S}(o) = \overline{\int_a^b} f(x)\,dx,$$

and since by hypothesis the upper and lower integrals are equal, it follows from the theorem[43]

If $\psi(x)$ and $\phi(x)$ have the same limiting value A as x approaches a, and if for all values of x in a sufficiently small neighborhood of a, we have

$$\psi(x) \leqq f(x) \leqq \phi(x),$$

then it follows that

$$\lim_{x \to a} f(x) = A.$$

Since the limits of $\lim_{x \to a} \psi(x)$, $\lim_{x \to a} \phi(x)$ exist and are equal to A, we have for x sufficiently near a

$$\psi(x) = A \pm e_1$$
$$\phi(x) = A \pm e_2$$

where e_1, e_2 are arbitrarily small positive numbers. Hence, for values of x sufficiently near a, we have

$$A \pm e_1 \leqq f(x) \leqq A \pm e_2,$$

or

$$e_1 \leqq |f(x) - A| \leqq e_2.$$

Since this inequality holds for all values of x sufficiently near a, say for x such that $|x - a| < d$, it follows that

$$|f(x) - A| < e \text{ for } |x - a| < d,$$

where e is an arbitrarily small number, and hence

$$\lim_{x \to a} f(x) = A.$$

that the limit

$$\lim_{\Delta \to 0} S(o)$$

exists and f(x) is integrable, the integral $\int_a^b f(x)\,dx$ being the common value of the upper and lower integrals.

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43. Ibid, 90, 91.

The given condition is also necessary. To show this, we have the condition that the limit

$$\lim_{\Delta \to 0} \sum_{k=1}^{n} f(E_k)\, \Delta_k x.$$

exists for all methods of subdivision of the given interval and for all values of E_k in $\Delta_k x$. It follows that the limit still exists if $f(E_k)$ is replaced by the least upper bound L_k or by the greatest lower bound l_k of $f(x)$ in $\Delta_k x$. When these values are substituted, we have the upper and lower integrals $\overline{\int_a^b} f(x)\, dx$, $\underline{\int_a^b} f(x)\, dx$, respectively, and their existence and equality follow as a consequence of the fact that they are special cases of the given limit.

Theorem (a) could be stated a little differently by adding several definitions. The difference between the maximum and minimum of a function $f(x)$ in an interval (a,b), is called the oscillation of f in (a,b).[44] It cannot ever be negative. Let n be any division of (a,b) into subintervals of d_k, of length d_k. Let w_k be the oscillation of f in d_k. The sum

$$\sum w_k\, d_k = \Omega_n(f,\ ab)$$

is called the oscillatory sum of f for the division n.

We have
$$\Omega_n(f,\ ab) = \sum_{k=1}^{n} (L_k - l_k)\, \Delta_k x$$

III - 7 $$= \overline{S}(o) - \underline{S}(o).$$

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44. Pierpont, Lectures on the Theory of Functions of Real Variables, I, 341, 342.

Theorem (b). In order that the limited function of f(x) be integrable in (a,b), it is necessary and sufficient that

III - 8 $$\underset{\Delta \to 0}{L} \Omega_n (f, ab) = 0.$$

For by III - 7 $\Omega_n (f, ab) = \overline{S}(o) - \underline{S}(o)$.

By theorem (a), f(x) is integrable only when

$$\underset{\Delta \to 0}{L} \underline{S}(o) = \underset{\Delta \to 0}{L} \overline{S}(o)$$

or when and only when

$$\underset{\Delta \to 0}{L} \overline{S}(o) - \underset{\Delta \to 0}{L} \underline{S}(o) = 0,$$

which proves III - 8.

Theorem (c). Given a function f(x) which is bounded in the interval (a,b). A necessary and sufficient condition that the R-integral of f(x) exists in (a,b) is that this interval may be divided into partial intervals such that the sum of the lengths of those subintervals in which the oscillation of f(x) is equal to or greater than any arbitrarily chosen positive number N may be made as small as one pleases.

Denote by S'_m the sum of the lengths of the subintervals of (a,b) in which the oscillation of f(x) is equal to or greater than N, and let $C(S'_m)$ be the sum of the lengths of the remaining subintervals of (a,b); that is, the sum of the lengths of those subintervals in which the oscillation is less than N. We may then write

III - 9 $$\Omega_n [f, ab] < N(b - a) + S'_m \cdot w,$$

where w is the greatest value of the oscillation w_k of f(x) in those subintervals included in S'_m. The value of w is finite and cannot increase as the size of the intervals is decreased by making the norm Δ to approach zero. We have given

$$S'_m < e,$$

where e is positive and may be chosen arbitrarily small. We then have from III - 9

$$\Omega_n [f, ab] < N(b - a) + e_1,$$

where e_1 is arbitrarily small. Since this relation holds for every arbitrary choice of N and e_1 however small, we have

$$\underset{\Delta \to 0}{L} \Omega_n [f, ab] = 0.$$

and f(x) is integrable by theorem (b).

To show that this condition is also necessary, we assume that f(x) is integrable in (a,b), and hence we have from theorem (b)

III - 10 $$\underset{\Delta \to 0}{L} \Omega_n [f, ab] = 0.$$

However we may write

$$\Omega_n [f, ab] = \sum_{k=1}^{n} w_k \Delta_k x \geqq N.S'_m + w.C(S'_m) \geqq N.S'_m$$

where w is the smallest value of w_k for the n subdivisions of (a,b). Substituting III - 10 in the last equation, we have, by passing to the limit,

$$\underset{\Delta \to 0}{L} \Omega_n [f, ab] = 0 \geqq N. \underset{\Delta \to 0}{L} S'_m$$

While N is arbitrarily chosen, it is greater than zero. Since both N and S'_m are positive we must have

$$\underset{\Delta \to 0}{L} S'_m = 0;$$

that is, we have

$$S'_m < e,$$

where e is arbitrarily small.

Theorem (d). The necessary and sufficient condition that a bounded function may be integrable (R), in the interval for which it is defined, is that the points of discontinuity of the function form a set of measure zero.[45]

It is convenient to express this condition in the form that the function must be continuous almost everywhere in the interval.

To show that the condition is necessary, let us consider the closed set G_k at which the saltus $w(x)$, of $f(x)$ is $\geqq k$, where k is a positive number. If an interval d contain a point of G_k within it, the fluctuation of $f(x)$ in d is $\geqq k$. If a point of G_k is the common end-point of two intervals, of equal length, the fluctuation of $f(x)$ in one at least of these intervals is $\geqq \frac{1}{2}k$; hence the part which these two intervals contribute to the sum $\sum dF(d)$ is $\geqq \frac{1}{2}kd$. If we have a net with equal meshes fitted on to (a,b), the contribution of all those meshes

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45. Hobson, op. cit., I, 465, 466.

which contain, within them or at an end-point, a point of G_k, is not less than the product of $\frac{1}{4}k$ into the sum of the breadth of these meshes. Unless the content of G_k is zero, the sum of the breadths of these meshes is greater than some fixed positive number, for all the nets of a symmetrical system. It is therefore necessary for the existence of the R-integral that the content of G_k should be zero; and this must be the case for every positive value of k. The set of points of discontinuity of the function is the outer limiting set of G_{kn}, where k_n is a sequence of diminishing values of k that converges to zero. It follows that the set of points of discontinuity of the function must have measure zero.

To show that the condition is sufficient, we observe that, if the content of G_k is zero, all the points of G_k are contained within the intervals of a finite set the sum of whose lengths is $< e$. The intervals complementary to this finite set have a total measure $> b - a - e$, and at every point in each of them $w(x) < k$. In accordance with the theorem[46]

> If $f(x)$ is bounded in the interval (a,b), and if k be a number greater than the upper boundary of $w(x)$ in (a,b), there exists a positive number alpha, such that in every closed sub-interval in (a,b) of length not exceeding alpha, the fluctuation $f(x)$ is $< k$.

each of these complementary intervals can be divided into a

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46. Ibid., 311 et. seq.

number of parts, in each of which the fluctuation is $< 2k$. Let this be done for each of the complementary intervals. We now have a net fitted on to (a,b), such that the sum of the breadths of those meshes in which the fluctuation is $\geq 2k$ is $< e$.

For this net $\sum dF(d) < e(\overline{S} - \underline{S}) + 2k\,(b - a - e)$; and since k and e are both arbitrarily small, a net can be determined for which $\sum dF(d)$ has an arbitrarily small value. The condition of integrability is therefore satisfied if, for every value of k, G_k has content zero, that is, if the set of points of discontinuity of the function has measure zero.

Theorem (e). If $f(x)$ is continuous in the closed interval (a,b), then the integral $\int_a^b f(x)\,dx$ exists.[47]

Since (a,b) is a closed interval, $f(x)$ is bounded and the foregoing theorems apply. There are no points of discontinuity, so the conditions of Theorem (c) are satisfied and the theorem follows.

Theorem (f). A bounded function $f(x)$ having a finite or enumerably infinite number of discontinuities is R-integrable.

As previously proven, every enumerable set is of measure zero. It then follows from Theorem (d) that $f(x)$ is integrable. This theorem shows that a bounded function may be

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47. Townsend, op. cit., 213, 214.

discontinuous at the set of rational points and still be integrable, provided it is continuous at the irrational points.

Theorem (g). A function $f(x)$ of limited variation in the closed interval (a,b) is integrable in the Riemann sense.

The given function is bounded and by the theorem[48]

> The points of discontinuity of a function of limited variation form at most an enumerable set. A function of limited variation can have only ordinary discontinuities, and the points of a given interval (a,b) at which $f(x)$ has ordinary discontinuities form at most an enumerable set.

Its points of discontinuity form at most an enumerable set. Consequently by Theorem (f) it follows that $f(x)$ is integrable.

Theorem (h). A bounded function $f(x)$ having only ordinary finite discontinuities in the given interval (a,b) is R-integrable.

By the above-quoted theorem, it follows that the points where the given function has ordinary discontinuities form an enumerable set. From Theorem (f) it follows that $f(x)$ is integrable in the interval for which it is defined.

Theorem (i). If $f(x)$ be bounded and monotone in (a,b); then $f(x)$ is integrable in (a,b).[49]

If $f(x)$ is constant, the theorem is obvious. If we show that for each $e > 0$ there exists a division n for which

$$\Omega_n \ (f, ab) < e$$

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48. *Ibid.*, 134, 206.
49. Pierpont, *op. cit.*, I, 343-346.

then by the theorem that

> In order that the limited function of f(x) be integrable in (a,b), it is necessary and sufficient that, for each $e > 0$, there exists at least one division n for which
>
> $$\Omega_n (f, ab) < e$$
>
> This latter condition is necessary from Theorem (b). It is sufficient. For by III - 7, for the division n
>
> $$\overline{S}(o) - \underline{S}(o) < e$$
>
> By Theorem (a) $\overline{\int_a^b} f(x)\,dx - \underline{\int_a^b} f(x)\,dx < e$
>
> But $$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$$
>
> (Since, if we had two numbers A, B, that $|A - B| < e$ however small $e > 0$ is taken; then $A = B$; for if $A \neq B$, say $A > B$, then $A - B$ is a definite positive rational number, say D. But $|A - B|$ is not $< D$, which contradicts the hypothesis and $A = B$.)

Therefore by Theorem (a) f(x) is integrable. As a specific example, suppose f(x) is increasing. Let us divide (a,b) into equal intervals of length.

III - 11 $$d < \frac{e}{f(b) - f(a)} .$$

Then

$$\Omega_n (f, ab) = d\left[\{f(a_1)-f(a)\} + \{f(a_2)-f(a_1)\} + \dots + \{f(b)-f(a_{n-1})\}\right]$$
$$= d\{f(b) - f(a)\}$$
$$< e, \text{ by III - 11}$$

CHAPTER IV

The Lebesgue Integral

In Chapter III the author defined the Riemann Integral and proved some existence theorems for this type of integral. Riemann integrability of f(x) implies boundedness of f(x). To say that a function is Riemann integrable means that certain appropriately formed approximating sums have a limit which is called the integral. Many bounded discontinuous functions are Riemann integrable, but there are quite a number which are not Riemann integrable. For this reason the Lebesgue definition of an integral has been introduced.

The Lebesgue definition of an integral together with the definitions of Stieltjes, Young, and others, which will be discussed in Chapter V are of importance in scientific discussion because they admit a larger range of integrable functions than the Riemann definition which is commonly used in elementary analysis and in the applications to the physical sciences.[50] In some instances, the Riemann and Lebesgue integrals exist in the same interval while in others a Lebesgue

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50. Townsend, *op. cit.*, 198.

integral of a bounded function may exist in the interval while the Riemann integral does not exist. A theorem of this type, the converse of which is not true, will be given later in this chapter in connection with the comparison of the two types of integrals mentioned. In accordance with Lebesgue's definition then, functions which possess a definite integral, form a class of functions that are integrable in accordance with Riemann's definition.

The Lebesgue theory of integration has as its foundation the conception of the measure of a set of points, according to the Lebesgue interpretation. In Lebesgue integration the domain over which the integral is taken is divided into a number of measurable sets of points, having a certain property relative to the function to be integrated, and the integral is defined as the limit of a certain sum taken for all these measurable sets of points, as the number of sets is indefinitely increased.[51] The essential difference between the two definitions of the integral rests upon the difference between the two modes of dividing the domain of integration into sets of points.

A function $f(x)$, defined in any interval (a,b), is said to be measurable, provided that, for every value of A, the set of points x, of (a,b), at which $f(x) > A$, is a measurable

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51. Hobson, <u>op</u>. <u>cit</u>., I, 562-564.

set of points. A can be any real number. This definition is applicable, whether x be a point of a linear set, or a point $(x^1, x^2, \dots x^p)$, in any number p, of dimensions.

Theorem (1). If $f(x)$ be a measurable function, defined at each point of a given domain, the sets of points for which

$$A < f(x) < B; \quad A \leqq f(x) < B; \quad A \leqq f(x) \leqq B; \quad f(x) < A;$$
$$f(x) \leqq A$$

are all measurable, whatever real numbers A and B denote, provided $A < B$.

In the first place, the domain for which $f(x)$ is defined, and for which it has a definite value at such point, is measurable. If A were given values $-N_1, -N_2, \dots -N_n, \dots$ successively, of a sequence such that N_n increases indefinitely as $n \to \infty$. The set E_n, for which $f(x) > -N_n$, is measurable, by hypothesis, for every value of n. The domain for which $f(x)$ is defined is the outer limiting set of the sequence E_n, of measurable sets, and is therefore itself measurable. The set of points for which $f(x) \leqq A$, is relatively complementary[52] to

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52. All point sets e which we consider are supposed to lie on a finite interval ab. The sum $e_1 + e_2$ of two sets e_1, e_2 is the totality of their points, the difference $e_1 - e_2$ is the set of points which are in e_1 but not in e_2, and the product of e_1e_2 is the totality of points which e_1 and e_2 have in common. Addition and multiplication are commutative and associative, and satisfy the relations $(e_1 + e_2)e_3 = e_1e_3 + e_2e_3$, $(e_1 - e_2)e_3 = e_1e_3 - e_2e_3$. The complement Ce of a set e is the totality of points of the interval ab which are not in e. The difference and product of two sets e_1, e_2 are expressible in terms of addition and complements. For
$$C(e_1 - e_2) = Ce_1 + e_2, \quad Ce_1e_2 = Ce_1 + Ce_2.$$

the domain of the function to the measurable set for which $f(x) > A$. Therefore $f(x) \leqq A$ is measurable. If A_n is a monotone increasing sequence of numbers converging to A, then all the sets for which $f(x) \leqq A_n$ are measurable, and their outer limiting set, for which $f(x) < A$, is consequently measurable. Since the sets for which $f(x) < A$ and $f(x) \leqq A$ are measurable, it follows that the set for which $f(x) = A$ is measurable.

Because $f(x) < B$, and $f(x) < A$, are measurable, their difference, the set for which $A \leqq f(x) < B$, is also measurable. From these properties the other results in the theorem follow.

Theorem (2). A function $f(x)$ is measurable if the set of points x is measurable, for which $A < f(x) < B$, for every pair A, B, of real numbers which belong to a given set, everywhere dense in the indefinite interval $(-\infty, \infty)$. The given set may be taken to be enumerable.

Let A and B be any pair of real numbers such that $A < B$. The number A can be expressed as the upper limit of a sequence α_n of increasing numbers, all of which belong to the given set which is everywhere dense; and the number B can be expressed as the lower limit of a similar sequence β_n of diminishing numbers. The set e_n for which $\alpha_n < f(x) < \beta_n$ is measurable, for each value of n; the inner limiting set e_n, of the sequence, is the set for which $A \leqq f(x) \leqq B$; and this set is consequently measurable. Since this is the case whatever values A and B may have, it is seen that $f(x)$ is measurable.

Theorem (3). A function $f(x)$ is said to be measurable (B), if the set of points for which $f(x) > A$ is measurable (B) whatever value A may have.

The above proofs show that the sets for which

$A < f(x) < B$; $A \leqq f(x) < B$; $A \leqq f(x) \leqq B$; $f(x) < A$; $f(x) \leqq A$

are all measurable (B).

Note: When the exterior and interior measures of a set G or points in p dimensions, are equal to one another, the set G is said to be measurable, and the number $m_e(G) \equiv m_i(g)$ is defined to be the measure of G. When G is measurable its measure is denoted by $m(G)$. All sets which are shown to be measurable, are obtained from the single point, the single interval, or cell, open or closed, by taking the complements of the sets so obtained. All sets defined in this manner are said to be measurable (B), since they are the only kind of measurable sets contemplated by Borel in his original treatment of metric properties.[53]

Theorem (4). If $\phi_1, \phi_2, \ldots \phi_n$ be a finite set of functions that are measurable in a measurable domain G, linear, or of higher dimensions, and if $F(\phi_1, \phi_2, \ldots \phi_n)$ be a function that is continuous relatively to $(\phi_1, \phi_2, \ldots \phi_n)$, for all values of $\phi_1, \phi_2, \ldots \phi_n$, then $F(\phi_1, \phi_2, \ldots \phi_n)$ is measurable in the given domain.

First, let us assume that all the functions $\phi_1, \phi_2, \ldots \phi_n$ are bounded in the given domain for which they are defined; suppose their values to be in the interval $(-N, N)$. Let a net

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53. Hobson, op. cit., I, 174-179.

$(c_0, c_1, \ldots c_m)$ be fitted on to the linear interval $(-N, N)$ where $c_0 = -N$, $c_m = N$, and suppose the width of each mesh, $c_r - c_{r-1}$ to be less than the positive number Nu. Let the function ψ_s be defined, corresponding to each function ϕ_s $(s = 1, 2, 3, \ldots n)$ by the conditions $\psi_s = c_{r-1}$ at every point at which $c_{r-1} \leq \phi_s < c_r$, for $r = 1, 2, 3, \ldots m$, and $\psi_s = c_m$ where $\phi_s = c_m$. We then have

$$0 \leq \phi_s - \psi_s < Nu,$$

and the function ψ_s taking only the values in the finite set $c_0, c_1, \ldots c_m$, this function is measurable in the given domain.

Since $F(\phi_1, \phi_2, \ldots \phi_n)$ is continuous in the closed domain, $(-N, -N, \ldots; N, N, \ldots)$, we have $\left| F(\phi_1, \phi_2, \ldots \phi_n) - F(\psi_1, \psi_2, \ldots \psi_n) \right| < \epsilon$, if Nu be taken sufficiently small; the number ϵ being chosen arbitrarily. The function $F(\psi_1, \psi_2, \ldots \psi_n)$ has only a finite set of values, and is measurable. If U and L are its upper and lower boundaries, we have

$$L - \epsilon < F(\phi_1, \phi_2, \ldots \phi_n) < U + \epsilon$$

in the whole domain. If A and B are any two numbers in the interval (L, U), then the set of points for which $A < F(\psi_1, \psi_2, \ldots \psi_n) < B$ is measurable.

Now let ϵ have successively the values in a sequence ϵ_t which converges to zero, then there exists a corresponding sequence Nu_t, of values of Nu, which converges to zero.

The set of points E_t, for which $A < F(\psi_1, \psi_2, \dots \psi_n) < B$, is measurable, for each value of Nu_t, in Nu_t sequence. Each point of the set for which

$$A < F(\phi_1, \phi_2, \dots \phi_n) < B$$

belongs to all the measurable sets E_t, from and after some particular value of t. Since the complement of the set is measurable, the set itself is measurable.[54] Thus $F(\phi_1, \phi_2, \dots \phi_n)$ is measurable in the domain for which the functions are defined.

Now let the functions $\phi_1, \phi_2, \dots \phi_n$ be unbounded. We define ϕ_r^N by the conditions $\phi_r^N = \phi_r$, when $N \geqq \phi_r \geqq -N$; $\phi_r^N = N$, when $\phi_r > N$; and $\phi_r^N = -N$, when $\phi_r < -N$. From the proof above we see that $F(\phi_1^N, \dots \phi_n^N)$ is measurable. If N were given successively the values in a divergent sequence N_t of increasing numbers; each point of the set for which $A < F(\phi_1, \phi_2, \dots \phi_n) < B$ belongs to all the measurable sets for which $A < F(\phi_1^{Nt}, \dots \phi_n^{Nt}) < B$, from and after some particular value of t. Then the set is measurable and the theorem holds when $\phi_1, \phi_2, \dots \phi_n$ are unbounded.

General theorem (5). The sum, or the product, of any finite number of measurable functions, defined in a measurable domain of any number of dimensions, is a measurable function.

If all the functions $\phi_1, \phi_2, \dots \phi_n$ are measurable (B),

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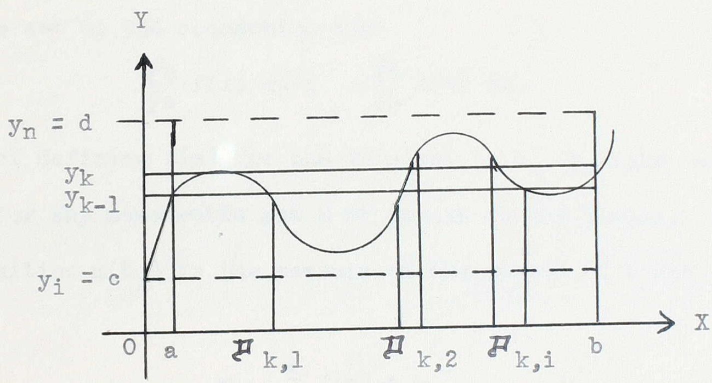
54. Ibid., 177, 178.

the function $F(\phi_1, \phi_2, \ldots \phi_n)$ is measurable (B), because all the sets used in above proofs are measurable (B).

This preliminary discussion brings us to the definition of the Lebesgue integral. Let $f(x)$ be a single-valued, bounded measurable function defined for the interval (a,b).[55] Let $y = c$ and $y = d$ be the lower and the upper bounds, respectively, of $f(x)$ in the given interval. Now instead of dividing the X-axis into subintervals as in the Riemann integral, we divide the interval (c,d) on the Y-axis into n subintervals by the insertion of the intermediate points $y_1, y_2, \ldots y_{n-1}$ as shown in Fig. X. Let E_k be the set of points in the interval

Fig. X.



(a,b) for which $y_{k-1} \leqq f(x) < y_k$. The Lebesgue integral may be considered as the measure of a two-dimensional point set. Let Nu_k denote any value of y in the interval (y_{k-1}, y_k). Al-

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55. Townsend, op. cit., 285-289.

so

IV - 1 $$\epsilon > (y_k - y_{k-1}),\ k = 1,\ 2, \ldots, n.$$

The Lebesgue integral may be analytically defined for the positive interval (a,b) by the relation

IV - 2 $$\int_a^b f(x)\, dx = \underset{\epsilon \to 0}{L} \sum_{k=1}^{n} Nu_k\, m(E_k),$$

where in this case,

$$m(E_k) = \sum_{i=1}^{p} D_k i.$$

As ϵ approaches zero, the value of n increased indefinitely although the converse may not be true.

To complete the definition for bounded, measurable functions, we set up the convention that

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx.$$

Instead of defining f(x) for the interval (a,b) it might be defined for any measurable set E of points on the X-axis. In the definition $m(E_k)$ is the measure of the subset of E for which

$$y_{k-1} \leqq f(x) < y_k$$

If the function f(x) satisfies this definition, it is said to be integrable in the Lebesgue sense over the set E, where

$$E \equiv \underset{\epsilon \to 0}{L} \sum_{k=1}^{n} E_k.$$

The Lebesgue integral will be indicated by the symbol

$$\int_E f(x)\, dx.$$

The Lebesgue integral is more effective than the integral of Riemann because the former may be associated in its applications with functions which are defined for a set of points instead of an interval.

It is not necessary that f(x) have only positive values as in the foregoing proof. All we need to do is to consider separately the set of points in the set x for which f(x) is negative and then take the algebraic sum of this result and the one for the positive values of f(x).

Theorem (6). The Lebesgue integral $\int_E f(x)\,dx$ exists if f(x) is measurable and bounded on the point set E.

We must show that the limit given in IV - 2 exists under the conditions set forth in the theorem and that the limiting value is independent of the manner in which the point set y_k is chosen. Let

IV - 3 $$\phi(n) \equiv \sum_{k=1}^{n} y_{k-1} m(E_k), \quad \psi(n) \equiv \sum_{k=1}^{n} y_k\, m(E_k).$$

Then IV - 4 $$\phi(n) \leqq \sum_{k=1}^{n} Nu_k\, m(E_k) \leqq \psi(n).$$

$\phi(n)$ is bounded and monotone increasing while $\psi(n)$ is bounded and monotone decreasing as the length of the intervals (y_{k-1}, y_k) decreases by inserting intermediate points. Consequently, each function has a limit as n increases indefinitely in such a manner that $\epsilon > y_k - y_{k-1}$ approaches zero. These limits are equal; for

$$\psi(n) - \phi(n) = \sum (y_k - y_{k-1}) \cdot m(E_k) < \epsilon \cdot m(E).$$

Since ϵ is arbitrarily small and E is measurable and hence m(E) is finite, it follows that $\epsilon . m(E)$ is arbitrarily small. We get

$$\text{IV - 5} \qquad \underset{\epsilon \to 0}{L} \left\{ \psi(n) - \phi(n) \right\} = 0.$$

By the theorem on page 31 of this thesis, it follows that IV - 4

$$\underset{\epsilon \to 0}{L} \sum_{k=1}^{n} Nu_k \, m(E_k)$$

exists as a limit and is equal to the common limit

$$\text{IV - 6} \qquad \underset{\epsilon \to 0}{L} \phi(n) = A = \underset{n \to \infty}{L} \psi(n).$$

It must yet be shown that the value of this limit is independent of the choice of the set of values y_k. Let y'_k be any other set of points in the interval (c,d). When the set y'_k is superimposed upon the set y_k, some of the points of the two sets coincide. The points of y'_k remaining may be regarded as the points inserted earlier in the discussion when the limits of IV - 6 were established. If the functions $\phi'(n)$ and $\psi'(n)$ be formed with reference to the set y'_n as we formed for y_k set, it follows that $\phi'(n), \psi'(n)$ will differ at most by $\epsilon' m(E)$, where ϵ' is arbitrarily small. This method may be continued by forming functions $\phi''(n)$ and $\psi''(n)$ with reference to the set y''_k and so on. As the functions approach the coincidence the functions $\phi'(n)$, $\psi'(n)$ must approach the same limiting value A. Since the value of the limit is not dependent upon the manner in which the set of points y_k was chosen, the theorem is established.

When it is known the integral exists, its value may be determined from either of the limits in IV - 6.

The above theorem proves the existence of the Lebesgue integral when f(x) is defined for all values of x in an interval (a,b), to be continuous throughout this interval and has only a finite number of minima and maxima. From IV - 2 we may see that the two integrals are equal. Nu_k can be replaced by $f(E_k)$, where $y_{k-1} \leqq f(E_k) \leqq y_k$, and in this case the measure $m(E_k)$ is the sum of the x-intervals composing E_k. [E in the Riemannian notation is really Epsilon; E was substituted to facilitate typing.] Upon substituting we can get

$$\underset{\Delta x \to 0}{L} \sum_{k=1}^{n} f(E_k)\, \Delta_k x = \underset{\epsilon \to 0}{L} \sum_{k=1}^{n} Nu_k m(E_k).$$

Thus for the special class of functions the Riemann and Lebesgue integrals are the same, and for bounded functions which are Riemann integrable there is a Lebesgue integral. The converse of this theorem is not true.

The Lebesgue double integral may be defined with respect to E by the relation

$$\iint_E f(x,y)\, dxdy = \underset{\epsilon \to 0}{L} \sum_{k=1}^{n} Nu_k m(E_k)$$

As a further comparison of the R and L integrals the following theorem might be stated.

Theorem (7). Let f(x) be defined for the interval (a,b) and for this interval suppose it to be single-valued, bounded, and integrable in the Riemann sense. It is then integrable in (a,b) in the Lebesgue sense and the two integral are equal. The converse, however, is not necessarily true.[56]

Because of limited space the proof for this theorem will be omitted. The reader will find this proof in the source recorded in the footnote. A brief consideration will be given to the converse of the theorem.

The converse of theorem (7) is not true. A Lebesgue integral of a bounded function may exist in an interval (a,b), while the Riemann integral does not exist. For example, let f(x) be defined for the interval (0,1) as follows:

$$f(x) = 1, \quad \text{for rational values of } x,$$
$$= 0, \quad \text{for irrational values of } x.$$

This function is totally discontinuous in the given interval and hence has no integral in the Riemann sense; because the necessary and sufficient condition of the R-integral is that the points of discontinuity shall form at most a set of measure zero. In this case the measure of the set of points of discontinuity is one. The function f(x) is bounded and measurable on the set E_1 of rational points and also on the set E_2 of irrational points. Consequently, by theorem (6) both the Lebesgue integrals $\int_{E_1} f(x)\, dx$, $\int_{E_2} f(x)\, dx$ exist. The Lebesgue integral $\int_0^1 f(x)\, dx$ taken over the interval (0,1) must exist because of the theorem[57]

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56. Ibid., 295-297.
57. Ibid., 290, 291.

If $f(x)$ is bounded and measurable on a finite number or an infinite sequence of distinct, measurable point sets E_n whose sum is E, then

$$\int_E f(x)\,dx = \int_{E_1} f(x)\,dx + \int_{E_2} f(x)\,dx + \cdots + \int_{E_n} f(x)\,dx + \cdots$$

This discussion would not be complete without mention of Lebesgue integrals for non-bounded functions.[58] Let $f(x)$ be any positive, measurable, non-bounded function defined on the bounded measurable set of points E. Let k be any one of the sequence of positive real numbers,

IV - 7 $$k_1, k_2, k_3, \ldots\ldots$$

having no upper bound. The auxiliary function $f_k(x)$ is defined

$$f_k(x) = f(x), \text{ where } f(x) \leqq k.$$
$$= k, \text{ where } f(x) > k.$$

Thus the function $f_k x$ is bounded and measurable on the set of points E. Consequently, the Lebesgue integral $\int_E f_k(x)\,dx$ exists for all values of k, Fig. XI. If the limit $\int_E f_k(x)\,dx$

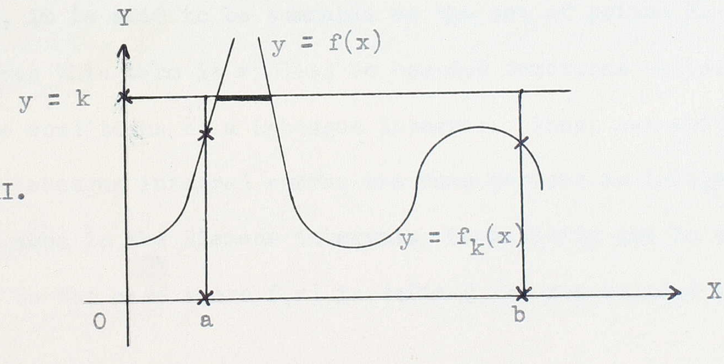

Fig. XI.

58. <u>Ibid.</u>, 297, 298.

exists as k becomes infinite, then we say that the integral $\int_E f(x)\, dx$ exists, and write

$$\int_E f(x)\, dx = \underset{k \to \infty}{L} \int_E f_k(x)\, dx$$

If f(x) is negative for all points of E, the Lebesgue integral exists on E if the limit $\underset{k \to \infty}{L} \int_E |f_k(x)|\, dx$ exists as k becomes infinite.

If f(x) is negative for some points of E and positive at others we may define the Lebesgue integral by the relation

$$\text{IV - 8} \qquad \int_E f(x)\, dx = \int_E f_1(x)\, dx - \int_E f_2(x)\, dx,$$

where $f_1(x) = f(x)$, if $f(x) \geqq 0$, $f_2(x) = f(x)$, if $f(x) < 0$,
$= 0$, if $f(x) < 0$, $= 0$, if $f(x) \geqq 0$.

It is assumed that the functions $f_1(x)$, $f_2(x)$ are measurable and that their integrals exist in accordance with the foregoing definition.

If the function f(x) satisfies the above definition IV - 8, it is said to be summable on the set of points E. Sometimes this term is applied to bounded functions satisfying the conditions of a Lebesgue integral. Thus, summability in the Lebesgue integral serves the same purpose as integrability does in the Riemann integral. Summability can be extended to the case where f(x) is defined for non-bounded sets E.

CHAPTER V

Other Modifications of the Definition of Integration

The definitionsof integration which have been suggested by Riemann and Lebesgue are the definitions which hold the central position in the theory of definite integration. These definitions have been considered in the last two chapters. Aside from these two definitions, there are various others. Some of these definitions are related to the R- and L-integrals, while others are equivalent to or extensions of these two definitions. In the present chapter the author will make brief mention of these modifications of the definition of integration. Some of the proofs for these definitions are too lengthy for this treatise. Where this is the case, the proof may be obtained by reference to the authority quoted.

W. H. Young gives the following definition of integration[59]

> Divide the interval (a,b) into a finite or a denumerably infinite number of measurable sets E_i of measure d_i. Let M_i be the least upper bound and m_i the greatest lower bound of $f(x)$ on E_i, and form the sums

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59. Hildebrandt, On Integrals related to and extensions of the Lebesgue Integrals, article in the Bulletin of the American Mathematical Society, Vol. 24 (1917-18), 120-123.

$$S = \sum_i M_i d_i \quad \text{and} \quad s = \sum_i m_i d_i ,$$

Then the greatest lower bound of S and the least upper bound of x for all possible divisions of (a,b) into measurable sets are defined to be the Young or Y upper and lower integrals of f(x) on (a,b). f(x) is said to be Y-integrable if the upper and lower integrals are finite and equal, that is, if

$$(Y)\int = (Y)\underline{\int} = (Y)\overline{\int}.$$

This definition is applicable in case the interval (a,b) is replaced by any measurable set of points E. The Y definition was originally suggested for functions f(x) bounded on (a,b), but the definition will apply if f(x) is not bounded, provided that f(x) is such that there exist partitions of (a,b) into a denumerable infinity of measurable sets on each of which f(x) has a finite upper and lower bound.

From the equation

$$(Y)\overline{\int} \geqq (Y)\underline{\int}.$$

it follows that if f(x) is Riemann integrable, it is also Y-integrable. Because it is possible to find a definition of the Lebesgue integral by replacing in the Darboux definition intervals by measurable sets, which Young did, we find the Young and Lebesgue definitions of integration are equivalent, and the values obtained by the two definitions are the same.

James Pierpont offers a definition of the integral which is an extension of the Lebesgue integral. In his definition Pierpont changes the definition of Lebesgue only in using an infinite number of cells instead of a finite number.

Lebesgue considered functions such that the points E at which a $\leqq f(x) \leqq$ b, for all (a,b) form a measurable set. He defined his integral as

$$\underset{\epsilon \to 0}{L} \sum_{k=1}^{n} Nu_k\, m(E_k),$$

where $y_{k-1} \leqq f(x) \leqq y_k$ in the set E_k whose measure is $m(E_k)$, and each $y_k - y_{k-1} \to 0$ as $\epsilon \to \infty$. Pierpont[60] has shown that if the metric field A be divided into a finite number of metric sets d_1, d_2, ... of norm D, then

$$\underline{\int}_A f = \mathrm{Max} \sum m_i d_i, \qquad \overline{\int}_A f = \mathrm{Min} \sum M_i d_i$$

where m_i, M_i are the minimum and maximum of f in d_i. If the cells d_1, d_2 ... are infinite instead of finite in number we get a theory of L-integrals which contains the Lebesgue integrals as a special case. The relation of the new integrals to the Riemannian integrals is obvious and the form of reasoning used in Riemann's theory may be taken over to develop the properties of the new integrals. Hildebrandt[61] gives a brief definition of the Pierpont integral. When the set E is measurable the definition is identical with that of Young.

When the number of points in the vicinity of which f(x) is not bounded, becomes infinite, then there are two types of definition.[62] One of them gives a definition by means of a

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60. Pierpont, op. cit., II, vi, vii.
61. Hildebrandt, op. cit., 127.
62. Ibid., 130, 131, et. seq.

single limiting process, the other by a denumerable set of such processes. The first of these leads to the Harnack-Jordan-Moore and Borel types of integration, the other to the Dirichlet, extended by Hoelder and Lebesgue, and Denjoy definitions of integration.

The set of points in every vicinity of which $f(x)$ is not bounded constitute a closed set. Harnack calls this set the set of singularities, z. The Harnack definition is as follows:

> Suppose the set z of singularities of $f(x)$ is of zero content. Enclose them in a finite set of intervals of total length ϵ. Let $f_1(x)$ be zero in the interior of the enclosing intervals, and equal to $f(x)$ everywhere else and suppose that $\int_a^b f_1(x)\,dx$ exists. If this integral approaches a finite limit as ϵ approaches zero, this limit is said to be the integral of $f(x)$ from a to b.

Jordan has a definition equivalent to that of Harnack in case the content of z is zero. The Jordan definition is:

> Divide (a,b) into any finite number of intervals of maximum length d. Exclude the intervals containing points of the set Z, and suppose that the (Riemann) integrals of $f(x)$ exist on the remaining intervals. If the sum of these integrals approaches a definite limit when d approaches zero, this is defined to be the integral of $f(x)$ from a to b.

E. H. Moore observed that the Harnack definition could be applied when the set Z is replaced by another Z_0 containing it, and that the resulting integral is a function of the set Z_0. Also, that in case the set of singularities Z is nonexistent then the integral of $f(x)$ on the basis of the set

Z_0 is equal to the ordinary integral of f(x) if the set Z_0 is of content zero. For that reason it is desirable to restrict consideration of these integrals to sets Z of content zero.

Borel's definition of integration reads as follows:[63]

> f(x) is Borel integrable in case (a) there exists a set of singularities Z denumerable or even of measure zero, such that for every ϵ and for every set of intervals which has total length at most ϵ and is such that each interval of the set contains at least one point of Z, the Riemann integral of f(x) on the complementary set P_ϵ exists, and (b) these Riemann integrals approach a finite limit as ϵ approaches zero. This limit is the Borel integral of f(x) on (a,b).

Townsend[64] writes that the Borel definition of integration is more restricted in its applications for bounded functions than the Lebesgue integral and that this definition can be applied to non-absolutely convergent integrals for non-bounded functions, which is not the case with the Lebesgue integrals. Also that when both the Borel and Lebesgue integrals exist, they have the same value.

The Denjoy integral is a generalization of the Lebesgue integral. Before stating the integral one must state certain principles of construction which are as follows:[65]

> 1. In any subinterval (A,B) of (a,b) in which the given measurable function f(x) is summable in the Lebesgue sense the Denjoy integral shall be identical with that of Lebesgue. The same shall also be true on any perfect set.
>
> 2. If the integral $D\int_A^B f(x)\,dx$ is known for all val-

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63. Ibid., 201.
64. Townsend, op. cit., 331.
65. Ibid., 328-330.

ues of A' < B' contained in (A,B), then we have

$$D\int_A^B f(x)\,dx = \lim_{\substack{A'\to A\\ B'\to B}} D\int_{A'}^{B'} f(x)\,dx.$$

3. If the Denjoy integral is known for a finite number of consecutive intervals (A_1, A_2) (A_2, A_3) ... (A_{n-1}, A_n), then

$$D\int_{A_1}^{A_n} f(x)\,dx = D\int_{A_1}^{A_2} f(x)\,dx + \dots + D\int_{A_{n-1}}^{A_n} f(x)\,dx$$

4. Let E be a perfect set of points contained in a subinterval (A,B) of (a,b), and suppose that f(x) is summable on the set E. Let it be supposed that the D integral has been defined on every subinterval (A',B') of (A,B) which contains no points of E as inner points. Let (A_n', B_n') $n = 1, 2, 3 \dots$ be the set of intervals complementary to E with respect to (A,B). Denote by M_n the upper limit of $\left| D\int_{A'}^{B'} f(x)\,dx \right|$ for all intervals (A',B') in (A_n, B_n). It is assumed that the series $\sum M_n$ converges.

The Denjoy integral for (A,B) is defined by

$$D\int_A^B f(x)\,dx = \sum_{n=1}^{\infty} D\int_{A_n}^{B_n} f(x)\,dx + L\int_E f(x)\,dx.$$

The Denjoy integral is then said to exist for the (a,b) interval by application of the foregoing principles of construction, if f(x) satisfied the following conditions.

I. If E is any perfect set contained in (a,b), the points of E in whose neighborhood f(x) is not summable shall form a subset which in no portion of (a,b) is everywhere dense with respect to E. A function is not summable in the neighborhood of a point, if there exists no subinterval containing the point on which the function is summable.

II. If $D\int_{A'}^{B'} f(x)\,dx$, A' < B' is known for all values of A',B' in (A,B), then f(x) must be such that the limit

$$\lim_{\substack{A'\to A\\ B'\to B}} D\int_{A'}^{B'} f(x)\,dx \quad \text{shall exist.}$$

III. If E is a perfect set in no subinterval everywhere dense and if $D\int_{A_n}^{B_n} f(x)\,dx$ is known in those subintervals (A_n, B_n) which are free from points of E, then f(x) shall be such that the points of E in the neighborhood of which the series $\sum_{n=1}^{\infty} D\int_{A_n}^{B_n} f(x)\,dx$ does not converge absolutely shall not be everywhere dense with respect to E in any subinterval.

If f(x) satisfies these conditions, then by the aid of the foregoing principles of construction the Denjoy integral can be calculated by means of an enumerably infinite set of Lebesgue integrals and passage to the limit.

There exist functions which have no Lebesgue integral, but which have a Denjoy integral. The Denjoy integral of the absolute value of the function does not exist, that is, it is not absolutely convergent. For the Denjoy integrals, as in the case of the Lebesgue integrals, the indefinite integral of f(x) is continuous and has f(x) as a derivative with the exception of at most a set of points of measure zero.

The Stieltjes integral is the integral of a continuous function with respect to a monotone increasing function. Young[66] proves the following theorem and then states his definition of the Stieltjes integral as follows:

> Theorem. If an ascending sequence of simple L-functions and a descending sequence of simple U-functions have the same limiting function, the limit of their integrals is the same.[67]
>
> Suppose all the functions defined in the closed interval (a,b). The theorem is then an immediate consequence of the theorem of bounds.

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66. Young, The Theory of Integration, 23.
67. A U-function is a function which is upper semicontinuous in an interval. An L-function is a function which is lower semicontinuous in an interval. f(x) is upper semicontinuous at the point X_0 if, given any positive quantity e, there is an interval having X_0 as middle point throughout which if $f(X_0)$ is finite $f(x) \leqq f(X_0) + e$, while, if $f(x_0)$ is $-\infty$, $f(x) \leqq -1/e$. The same is true of lower semicontinuity with the inequality sign in the opposite direction and the signs of ∞ and 1/e positive.

The difference between corresponding functions form a monotone descending sequence of simple U-functions

$$c_1(x),\ c_2(x)\ \ldots$$

having the limit zero. By the theorem of bounds their upper bounds u_n also have the limit zero. From the property, if f_1 is greater than or equal to f_2 throughout (a,b), then

$$\int_a^b f_1\, dg \geqq \int_a^b f_2\, dg$$

it follows that

$$\int_a^b c_n(x).dg(x) \leqq U_n \cdot \int_a^b dg(x) \to 0.$$

The theorem follows because the integral of the sum of two functions is equal to the sum of their integrals.

Definition. Given any continuous function we can always construct a monotone ascending sequence of simple L-functions and a monotone descending sequence of simple U functions of which it is the limit. The limit of the integrals is, by the above theorem, independent of the choice of these sequences. (Given two pairs of sequences, we need only compare the U-sequence of one pair with the L-sequence of the other.) This limit is defined to be the integral of the given continuous function.

Stieltjes considered only the case of one variable. The method of monotone sequences is independent of the number of variables concerned. When the integrator is the product of these variables, area or volume, the definition reduces to that of the multiple integral used for special types of Cauchy and Riemann functions.

Frechet[68] suggested a definition of integration which includes the Lebesgue, Young, Pierpont, and Stieltjes integrals as special cases by properly assigning the function.

Hellinger's definition of an integral is closely related

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68. Hildebrandt, op. cit., 189-194.

to that of Stieltjes.[69] He considers two functions f(x), which is continuous in (a,b), and $\phi(x)$ which is continuous and monotone increasing. He further assums that the function f(x) is constant in any portion of (a,b) where $\phi(x)$ is constant. Then if $[\phi(x_2) - \phi(x_1)]$ is zero, $[f(x_2) - f(x_1)]$ is zero. He next divided (a,b) into a finite number of intervals

$$x_1, x_2, \ldots, x_{n-1},$$

and formed the sum

$$\sum_{k=1}^{n} \frac{[f(x_k) - f(x_{k-1})]^2}{\phi(x_k) - \phi(x_{k-1})}$$

the quotient being defined as zero whenever the denominator is zero. The least upper bound of this sum for all methods of subdivision of the given interval is called the Hellinger integral and is denoted by

$$\int_a^b \frac{[df(x)]^2}{d\phi(x)}$$

The Hellinger integral may be expressed in terms of a Lebesgue integral and conversely.

Radon has modified the definition of integration by a generalization of the Hellinger integral.

Perron introduces a different notion of integration when he considered integration as the inverse process of differentiation. He introduces the adjoined upper and lower functions and thus is able to formulate a definition of integration which

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69. Townsend, op. cit., 328.

for bounded functions is identical with the Lebesgue integral, but for non-bounded functions leads to a more general class of integrable functions. Under certain restrictions the Perron integral for non-bounded functions becomes identical with that of Lebesgue. The Perron function can be readily extended to functions of two or more variables.[70]

Another mode of defining the integral of a function in a finite interval has been developed by Tonelli. His method is independent of the general theory of the measure of sets of points.[71]

Authors differ as to their opinions regarding different methods of integration. Hobson[72] writes

> The Riemannian integral is not only of interest from a historical point of view, but it still possesses great intrinsic importance in Analysis, and will probably continue to be the basis upon which practical applications of the integral calculus rest.

Agnew says[73]

> The theory of Lebesgue integration, depending as it does on the theory of Lebesgue measure of point sets, appears to be more complicated than the theory of the definite integral of the elementary calculus, that is the Riemann integral. This is the only reason why the Lebesgue integral has not completely supplanted the integral of Riemann which de la Vallee Poussin[74] once described as having only historical interest. The Lebesgue integral is in fact a far more elegant and useful tool than that of Riemann. It is to be regretted that those who recognize the beauty and utility of the Lebesgue

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70. Ibid., 330, 331.
71. Hobson, op. cit., II, 380-382.
72. Ibid., I, 460.
73. Agnew, Convergence in Mean and Lebesgue Integration, article in the American Mathematical Monthly, Vol. XLIV, 4.
74. De la Vallee Poussin, Cours d'Analyse, 250.

integral have done so little to popularize it and its applications.

Bliss[75] seems to sum the argument correctly when he argues

> In the field of integration the classical integral of Riemann, perfected by Darboux, was such a convenient and perfect instrument that it impressed itself for a long time upon the mathematical public as being unique and final. The advent of the integrals of Stieltjes and Lebesgue has shaken the complacency of mathematicians in this respect, and, with the theory of linear integral equations, has given the signal for a reexamination and extension of many of the types of processes which Volterra calls passing from the finite to the infinite........The definitions of Lebesgue, Young and Pierpont, and those of Stieltjes and Hellinger, form two rather well defined and distinct types, while that of Radon is a generalization of the integrals of both Lebesgue and Stieltjes. The efforts of Frechet and Moore have been directed toward definitions valid on more general ranges than sets of points of a line or higher spaces, and which include the others for special cases of these ranges. Lebesgue and Hahn, with the help of somewhat complicated transformations, have shown that the integrals of Stieltjes and Hellinger are expressible as Lebesgue integrals. ....
>
> The conclusion seems to be that one should reserve judgment, for the present at least, as to the final form or forms which the integration theory is to take. It is probable that the outcome may be a general theory of the type of those of Frechet and Moore, having not one but a number of special instances with forms more adaptable to problems of various special types. However this may be, there can be no question as to the wide influence which the work of Borel, Lebesgue and their followers is having upon the mathematical thought of the present time, and no question as to the notable advances which have been made in the many domains of real function theory to which the Lebesgue form of integral is especially adapted.

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75. Bliss, Integrals of Lebesgue, article in the Bulletin of the American Mathematical Society, Vol. XXIV, 1-3.

CHAPTER VI

Summary

The Riemann or common integral implies boundedness of f(x). Riemann integrability means that certain appropriately formed approximating sums have a limit which is called the integral. Many bounded discontinuous functions are Riemann integrable, but there are quite a number which are not Riemann integrable. Essentially Riemann integration means summing along the X-axis.

Where there are bounded discontinuous functions which are not Riemann integrable, the Lebesgue integral is used. Lebesgue integration has as its foundation the conception of the measure of a set of points and is taken along the Y-axis. The domain over which the integral is taken is divided into a number of measurable sets of points, having a certain property relative to the function to be integrated, and the integral is defined as the limit of a certain sum taken for all these measurable sets of points, as the number of sets is indefinitely increased. The essential difference between the Riemann and Lebesgue definitions rests upon the difference between the two modes of dividing the domain of integration into

sets of points. Lebesgue integrals besides being applicable to bounded discontinuous functions, exist for unbounded functions. In neither of these two cases are Riemann integrals applicable. In accordance with the Lebesgue definition functions which possess a definite integral, form a class which is markedly wider than, and includes the class of functions that are integrable in the Riemann sense.

Modifications of the definitions of integration just mentioned, which have been discussed in Chapter V are of importance in scientific discussion because they admit a larger range of integrable functions than the Riemann and Lebesgue definitions. In most cases, however, they lead to results which would have been the same had they been obtained by means of these two commoner definitions. There are many cases which could not be solved by the Riemann and Lebesgue integrals because of certain specifications introduced. Often a transformation in the problem will change the modification of the definition to a problem which is Riemann or Lebesgue integrable.

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