Spring 1935

\( e \)

Doyle K. Brooks

Fort Hays Kansas State College

Follow this and additional works at: https://scholars.fhsu.edu/theses

Part of the Algebraic Geometry Commons

Recommended Citation

https://scholars.fhsu.edu/theses/232

This Thesis (L20) is brought to you for free and open access by the Graduate School at FHSU Scholars Repository. It has been accepted for inclusion in Master's Theses by an authorized administrator of FHSU Scholars Repository.
A thesis presented to the Graduate Faculty
in partial fulfillment of the
requirement for the
degree of
Master of Science

by

Doyle K. Brooks, A. B., (B. S. in Ed.)

FORT HAYS KANSAS STATE COLLEGE

1935

Approved by:

[Signature]
Professor of Mathematics

[Signature]
Chairman of the
Graduate Council
CONTENTS

INTRODUCTION

CHAPTER

I.  e AND A NATURALIST  

II. A BRIEF HISTORY OF THE THEORY OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS  

III. e AND THE LOGARITHM  
The Logarithmic Concept  
Logarithmic Tables  
Modulus of Common Logarithms  

IV. THE LOGARITHMIC SPIRAL  

V. THE CATENARY  

VI. HYPERBOLIC FUNCTIONS  
Analytical Definition  
Geometrical Definition  
Applications of Hyperbolic Functions  

VII. THE TRANSCENDENTALISM OF e  

VIII. APPLICATIONS OF e  
Properties of e  
e as a Logarithmic Base  
The Catenary  
The Law of Exponential Growth  
Applications of the Derivative of e^x  

BIBLIOGRAPHY
ACKNOWLEDGEMENTS

The writer of this thesis wishes to acknowledge all sources of information used in its preparation. He is especially indebted to the following members of the faculty of the Fort Hays Kansas State College for their invaluable aid and counsel: Professor E. E. Colyer, Dr. W. G. Warnock, Dr. H. A. Zinszer, Dr. F. B. Streeter, Dr. R. R. Macgregor, and Dr. Floyd B. Lee. He is indebted to Dr. Wealthy Babcock and Miss Mary Strain of the University of Kansas. He also wishes to thank the librarians of the Fort Hays Kansas State College, the Kansas State College of Agriculture and Applied Sciences, and the University of Kansas.
The meteoric splendor of the transcendental constant $e$ is an intriguing mystery. It rises unexpectedly from nowhere, brilliantly illuminates some obscure mathematical concept, points to its solution, and abruptly fades into oblivion. Its unheralded visitation leaves the student wiser but wondering, tantalized by its omnipotence in apparently unrelated fields.

Ever since his first introduction to $e$ in elementary logarithms it has seemed to the writer that all the authors of textbooks are in a conspiracy to defeat any real knowledge of $e$. They say "$2.71828\ldots$, called $e$, is the base of Naperian logarithms." Why? "The derivative of $e^x$ is $e^x$." Why? "Placing $y = e^x$, the solution is evident." Why? To answer these interminable "why's" is the object of this thesis; to conquer this phantom in a field of logic.

The laconic title, $e$, is perhaps too all-inclusive. It tends to connote an absolute knowledge of everything pertaining to $e$. The writer does not pretend to that knowledge. Rather he has attempted
to gather from all available sources a little of everything concerning this constant. In this attempt he hopes to justify the title e.

In his 'Entomologiques', a delightful picture of the harmony between rigorous logic and nature. The writer will let H. Fabre present this picture in his own inimitable manner.

Let us direct our attention to the webs of Epeirea, preferably to those of the silky Epeirea and the striped Epeirea, numbers in autumn in my neighborhood, and so noticeable by their size.

We shall observe that in each sector, the various steps or elements of each turn of the spiral, are parallel to one another, and close gradually upon one another as they near the centre. They make, with the two radii which limit them at either end, an obtuse and an acute angle, on the side away from, and towards the centre, respectively, and these angles are the same throughout the same sector, because of the parallel disposition of these elements of the spiral.

More than this: in different sectors these obtuse and acute angles are the same, as far as one can rely on the testimony of the eye unaided by any measuring instrument. In a whole, the funicular structure is therefore a series of transverse threads which cut obliquely the various radii at an angle of invariable magnitude.

This is the characteristic feature of the logarithmic spiral. Geometers give this name

\[1\text{De Bray, M. E. J. C. Exponentials Made Easy, pp. 2-12 (Translated from the French by the author)}\]
CHAPTER I

e AND A NATURALIST

Henri Fabre, a rare combination of mathematician and entomologist, has given us, in his Souvenirs entomologiques, a delightful picture of the harmony between rigorous logic and nature. The writer will let H. Fabre present this picture in his own inimitable manner.

Let us direct our attention to the webs of Epeirae, preferably to those of the silky Epeira and the striped Epeira, numerous in autumn in my neighborhood, and so noticeable by their size.

We shall observe that in each sector the various steps or elements of each turn of the spiral, are parallel to one another, and close gradually upon one another as they near the centre. They make, with the two radii which limit them at either end, an obtuse and an acute angle, on the side away from, and towards the centre, respectively, and these angles are the same throughout the same sector, because of the parallel disposition of these elements of the spiral.

More than this: in different sectors these obtuse and acute angles are the same, as far as one can rely on the testimony of the eye unaided by any measuring instrument. As a whole, the funicular structure is therefore a series of transverse threads which cut obliquely the various radii at an angle of invariable magnitude.

This is the characteristic feature of the logarithmic spiral. Geometricians give this name

1De Bray, M. E. J. G. Exponentials Made Easy, pp. 2-12 (Translated from the French by the author).
to the curve which cuts obliquely, at a constant angle, all the straight lines radiating from a centre, called the pole. The web of Epeira is nothing else but a polygonal line inscribed in a logarithmic spiral. It would coincide with this spiral if the radii were unlimited in number, so that the rectilinear elements were indefinitely short and the polygonal line modified into a curve.

The Epeira winds its thread nearer and nearer to the pole of its web as closely as it is enabled to do so by the imperfection of its tools, which, like ours, are inadequate to the task; one would think that it is deeply versed in the properties of the spiral.

H. Fabre then relates several peculiar properties of the logarithmic spiral, including the evolute developed by Jacques Bernouilli, and the rectilinear trajectory which is the locus of the center of the spiral as the spiral is caused to roll upon a straight line. He continues:

Is this logarithmic spiral, with its curious properties, merely a conception of the geometers, who combine number and space at will to open a field wherein to practice mathematical methods? Is it but a dream in the night of the intricate, and abstract enigmas intended to feed our understanding? Not at all. . . . . It is a reality in the service of life. . . . a plan of which animal architecture makes frequent use. The mollusc, in particular, never shaped the volutes of its shell without reference to this transcendent curve. The first-born of the series knew it, and copied it, as perfect in primaeval times as it is today.
What is its guide? Necessarily, the animal must have in itself the virtual design of its spiral. Never could chance, however fecund in surprises we suppose it to be, have taught it the high geometry where our mind goes astray without a preliminary training.

Can it be premeditated combination on its part? Is there calculation, mensuration of angles, verification of parallelism, by sight or otherwise? I incline to believe that there is nothing of all that... nothing but an innate propensity of which the animal has not to regulate the effect, no more that the flower has to regulate the disposition of its petals. The spider practises advanced geometry without knowing, without caring.

The process goes by itself, the initial impulse having been given by an instinct conferred at the origin.

The pebble thrown by the hand, in returning to the ground, describes a certain curve; the dead leaf, detached and carried away by the wind, in performing its journey from the tree to the soil, follows a similar curve. In either case no influence of the moving body regulates the fall... nevertheless, the descent is performed according to a scientific trajectory, the parabola, of which the section of a cone by a plane has furnished the prototype for the meditation of geometers. A figure, the fruit of a speculative concept, has become tangible by the fall of a stone out of the vertical.

The same speculations take up the parabola once more and suppose it to roll on an indefinite straight line, and enquire into the nature of the path followed by the focus of the curve. The reply is that the focus of the parabola describes the catenary, a line of very simple shape, but for the algebraical expression of which we must have recourse to a cabalistic number, at variance with all systems of numeration, and which digits refuse to express exactly, however far one may pursue their orderly array. This number is called \( \epsilon \), being represented by the Greek letter \( \epsilon \).

Most authors designate this constant by \( e \) rather than \( \epsilon \).
H. Fabre then defines the value of this constant as the summation of the exponential series, and gives its numerical value as 2.7182818....

With this strange number, are we now restricted to the rigid domain of the mind? Not at all: the catenary appears in the realm of reality whenever gravitation and flexibility act jointly. This name is given to the curve formed by a chain suspended by two points not situated on the same vertical. It is the shape naturally taken by a flexible tape the two ends of which are held in one's hands; it is the outline of a sail inflated by the wind; it is the form of the milk-bag of the goat returning from the pasture where its udder became filled.... and all these things involve the number epsilon.

What a lot of abstruse science for a bit of string! Let us not be surprised. A pellet of lead swinging at the end of a thread, a drop of dew trembling at the end of a straw, a puddle ruffled by ripples under a puff of air, a mere nothing, after all, requires a titanic scaffolding when we wish to examine it with the eye of the calculus. .... We need the club of Hercules to crush such a midget!

Surely our methods of mathematical investigation are full of ingenuity. .... one cannot admire too much the powerful brains which have invented them. .... but how slow and painstaking when facing the least realities! Shall it ever be given to us to investigate the truth in a more simple fashion? Shall mind be able some time to do without the heavy arsenal of formulæ? Why not?

Here the occult number epsilon reappears inscribed on a spider's thread. On a misty morning, look at the web which has just been constructed during the night. Owing to its hygroscopic nature the sticky network has become laden with droplets, and, bending under the weight, the threads are now as many catenaries, as many rosaries of limpid gems, graceful rows of beads, arranged
in exquisite order, and hanging in elegant curves. Let the sun pierce the mist... and lo! the whole becomes iridescent with adamantine fire and, in lovely garlands of fairy lights, the number e appears in all its glory!

Geometry, that is, the science of harmony in space, presides over all things. It is in the arrangement of the scales of a fir-cone, as in the disposition of the Epeira's web: it is in the shell of a snail as in the rosary of a spider's dewladen thread, as in the orbit of a planet; it is everywhere, as majestic in an atom as in the world of immensities...

And this Universal Geometry speaks to us of a Universal Geometer, whose divine compass has measured all things... As an explanation of the logarithmic curve of the Ammonite and of the Epeira, it is perhaps not in agreement with the teachings of to-day... but how much loftier is its flight!

---

CHAPTER II

A BRIEF HISTORY OF THE THEORY OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

The elementary operations of multiplication and division presented extraordinary difficulty to Europeans of the Middle Ages. With their clumsy, inefficient Roman notation, it was necessary to resort to a glorified system of finger counting, the abacus. Not until the close of the fifteenth century was the Arabic, or, more properly, the Hindu, notation, widely accepted, although it had been perfected in the ninth century.

Even after the new notation, aided by decimal fractions, had greatly simplified these processes, it was found that a demand had arisen for operations with many large factors. This demand was the result of the new science heralded by Kepler and Galileo, who were attempting to measure stellar magnitudes. And, as a gift of the magi, the demand was met. Logarithms, the last of the trinity that has made possible the miraculous

---

powers of modern calculation, were born into an awaiting world.²

An obscure German mathematician, Michael Stifel, whose *Arithmetica Integra* appeared in 1544, developed what is perhaps the first rudimentary logarithmic table. It contains only the integers from -3 to 6 as exponents of 2, together with the corresponding powers 1/8 to 64. It appears that he saw intuitively the significance of his discovery, for he writes that one might devote an entire book to these remarkable number relations.³

Stifel was handicapped by the lack of two workable notations—exponents and decimal fractions—although he himself made an unsuccessful attempt to introduce an exponential notation.⁴

To Simon Steven of Bruges in Belgium (1548-1620) belongs the honor of the invention of decimal fractions. In 1585 he published his *La Disne*, containing seven pages, in which he applied decimal fractions to the operations of ordinary arithmetic.⁵ But it was not

²Ibid., p. 155.
⁴Cajori, op. cit., p. 157
⁵Ibid., p. 151.
until after 1600 that decimal fractions became common property of calculators.

Consider the situation at the time of the invention of logarithms in 1614. Scientists were demanding simplified arithmetic methods. Arabic notation had been greatly improved by the decimal fraction, but a satisfactory conception of the exponent, upon which the modern explanation of logarithms depends, was not to be developed for many years. Incredibly, at this point, John Napier, or Neper, a Scotch nobleman, gave his great invention of logarithms to posterity. Lord Moulton says of Napier's invention:

The invention of logarithms came on the world as a bolt from the blue. No previous work had led up to it, foreshadowed it or heralded its arrival. It stands isolated, breaking in upon human thought abruptly without borrowing from the work of other intellects or following known lines of mathematical thought.6

His invention was improved and extended by himself, Henry Briggs, Adrian Vlacq, Edmund Gunter, Henry Gellibrand, John Speidell, and others.7

Joost Bürgi, a Swiss, developed a system of logarithms contemporaneously but independently of Napier, and published his results in 1620.8

---

7Cajori, op. cit., p. 167.
8Ibid., p. 166.
Nicolaus Mercator (1620-1687) was the first to conceive the logarithm as the area under the hyperbola. He also introduced the name "natural logarithm", used in its present sense. His greatest achievement was the setting up of the power series which he obtained from the integral representation of the logarithm by dividing the numerator by the denominator and integrating term by term, thus developing the series:

$$\log (1 + x) = \int_0^x \frac{dx}{1 + t} = \int_0^x (1 - x + x^2 - x^3 + \ldots)dx$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$$

The results of Mercator were the inspiration of Newton, who enriched them with the binomial theorem and the reversion of series. By reversion of series he derives the exponential series:

$$x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

He writes, "Ponamus autem hanc serie..." from Mercator's series for $y = \log x$. This yields, as the number whose natural logarithm $y = 1$

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$$

From this we may show that for every real rational $y$,
x is one of the values of $e^y$. The function $y = \log x$ becomes the logarithm of $x$ to the base $e$, where $e$ is defined by means of the above series. Brook Taylor amplified this chain of discoveries in his general principle for developing functions into power series.\textsuperscript{11}

Leonhard Euler (1707-1783), in his *Introductio in analysin infinitorum*, 1748, expands $(1 + 1/n)^ny$ by the binomial theorem, obtaining the series

$$(1 + 1/n)^{ny} = 1 + n y \frac{1}{1 \cdot n} + n y(ny - 1) \frac{1}{1 \cdot 2} \frac{1}{n^2} + \ldots$$

He then takes the limit of each term, and determines a value

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$$

To Euler is due the use of the algorithm $e$ to represent the limit of $(1 + 1/n)^n$ as $n$ increases without bound. He writes, "Ponamus autem brevitatis gratia pro numero hoc 2.71828... constanter litteram $e$"; ("But we may place, for brevity, $e$ in place of the constant number 2.71828...")\textsuperscript{12}. Boorman calculated $e$ to 346 decimal orders.\textsuperscript{13}

Tests for rigor came with the nineteenth century.

\textsuperscript{11} Ibid., pp. 151-152.
\textsuperscript{12} Ibid., p. 152.
\textsuperscript{13} White, W. F. *Scrap Book of Mathematics*, p. 40.
Gauss, Abel, Cauchy, and others undertook to determine the convergence of infinite series and other infinite processes. The result of their research, for the series under consideration, is that all the earlier developments are correct under certain conditions, although the rigorous proofs are very complicated.\textsuperscript{14}

By means of the infinitesimal calculus, firmly entrenched by Cauchy, the theory of the logarithm was established with full mathematical exactness.

The theory of the functions of the complex variable, involving logarithmic and exponential functions, was first comprehended by Gauss, but elaborated by Cauchy.

This brief summary is necessarily confined to mathematicians of the foremost rank, although many others have contributed to the theory.

\textsuperscript{14} Ibid., p. 154
CHAPTER III

e AND THE LOGARITHM

The Logarithmic Concept

The word "logarithm", which means "ratio number" from the Greek λόγος (log'os), ratio, plus ἀριθμός (arithmos'), number, was created by Napier.¹ Since Napier is so intimately connected with the theory of logarithms, it will be interesting to follow his train of thought, even though the modern conception, based on exponential notation, is radically different from his original presentation.

Napier was primarily interested in multiplications involving sines, and it was only later, under the influence of Briggs, that he broadened the field to include numbers generally.² At that time the sine of an angle was not regarded as a ratio, but as the length of that semi-chord of a circle of given radius which subtends the angle at the center. Napier took this radius to be 10⁷ units; and thus the sine of 90° is 10⁷ and the sine of 0° is 0. Therefore his original table is one of logarithms of numbers between 10⁷ and 0, not for equidistant numbers, but for numbers corresponding

²Smith, David Eugene. Sourcebook in Mathematics, p. 149.
to equidistant angles.

If any two series, one geometric and the other arithmetic, are compared, it is found that sums in the arithmetic series correspond to products in the geometric series. Thus, in the relationship between the two series,

\[
\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8, 16, 32, 64, \ldots
\]

\[
-3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots
\]

mentioned by Stifel, (see Chapter II, p. 7) if we add any two terms in the lower series, the sum is the exponent of the constant ratio, 2, which expression is equivalent to the product of the corresponding terms in the upper series. We may denote the terms of the lower series as logarithms of the terms in the upper series, to the base 2.

Napier had no explicit knowledge of an exponent, of e, nor of a base, although he realized the arbitrary element in logarithmic systems. Instead, he conceives the motion of two points moving on parallel lines, the one with a constant velocity and the velocity of the other proportional to its distance from its goal.

Consider the straight line TS (Fig. 1, p. 14) to be
10^7 units in length. As the point P moves from left to right, its velocity at every point is proportional to its distance from S.\(^4\)

Fig. 1

On another straight line of indefinite length, Q moves with uniform velocity equal to that of P when P is at T. Consider Q to be at T\(_1\) when P is at T. Napier defines the logarithm of the sine, or the length SP\(_1\) when P is at P\(_1\) as the number representing the length T\(_1\)Q\(_1\), which is the distance Q has moved while P has progressed to P\(_1\). The logarithm of the sine of 90°, or 10^7, is 0, and the logarithm of any sine less than 10^7 is positive and increases as the sine diminishes to zero. The logarithm of any number greater than 10^7 is negative.

Napier constructed his tables by determining two limits between which a logarithm must lie. His whole method of construction is based upon the use of these limits.\(^5\)

\(^4\)Ibid., p. 23.
\(^5\)Ibid., p. 27.
The velocity of P at T equals the velocity of Q at T1; the velocity of P decreases thereafter, while that of Q remains constant; the velocity of P at p1 is greater than the velocity of Q at q1. Therefore,

\[ TP_1 < T_1 Q_1, \quad P_1 T > q_1 T_1. \]

If \( X = P_1 S, N \log X = T_1 Q_1, \) and

\[ N \log x > TP_1, \] or \( 10^7 - x. \]

Also, \( N \log x = q_1 T_1 < P_1 T, \) or \( TP_1 10^7/x, \)
which is \( (10^7 - x)10^7/x. \)

Therefore,

\[ (10^7 - x)10^7/x > N \log x > 10^7 - x. \]

These are Napier's limits for a logarithm. In a similar manner it may be shown that

\[ 10^7(y - x)/x > N \log x - N \log y > 10^7(y - x)/y. \]

In the absence of suitable algorithms, Napier stated the above results in words.

---

6Ibid., p. 28.

Klein, Felix. Elementary Mathematics from an Advanced Standpoint, p. 146.
At this point we will attempt to show how the logarithms developed by Napier and Bürgi, whose method of procedure was somewhat similar to Napier's are related to $e$, the base of natural logarithms.

In both systems we may start with the exponential equation, $x = b^y$. The object is to find for every $x$ a logarithm, $y$. Today this is achieved by fractional exponents, but Napier and Bürgi intuitively realized that the same result may be approximated by making the value of $b$ conveniently close to 1, and using integers for $y$. Bürgi takes $b = 1.0001$, while Napier's value, $b = 1 - 0.0000001 = .9999999$, is less than one, it is still closer to one than that of Bürgi. We will confine the rest of this discussion to Bürgi's value, 1.0001.

First are calculated the powers for two neighboring exponents, $y$ and $y + 1$:

$$x = (1.0001)^y, \quad x + \Delta x = (1.0001)^{y + 1}.$$  

Subtracting,

$$\Delta x = (1.0001)^y (1.0001 - 1) = x/10^4,$$

or, writing $\Delta y$ for the differences, 1, of the values of the exponent,

$$\frac{\Delta y}{\Delta x} = \frac{10^4}{x}.$$

---

After Bürgi had obtained the $x$ corresponding to a certain $y$, he determined the following $x$ belonging to $y + 1$ by the addition of $x/10^4$.

Dividing by $10^4$, the result is a simplified difference equation,

\[ \frac{\Delta y}{\Delta x} = \frac{1}{x} \]

in which the increment of $y$, $1$, is supplanted by $.0001$.

We will now attempt to show that the Bürgi logarithm may be represented as the summation of a series of rectangles under a curve. Since the logarithm increases arithmetically by the addition of $10^{-4}$, it follows that each increment in area is equal to the one preceding, and is equal to $10^{-4}$. From the difference equation (1),

\[ \Delta y = \Delta x \cdot \frac{1}{x} \]

and, since we are considering $\Delta y$ as the area of a rectangle, $\Delta x$ may be considered its base, and $1/x$ its altitude. But, since the $x$ increments increase in such a way that

\[ y_{i+1} = y_i + \Delta y_i, \]

where $\Delta y_i = 1/10^4$, the $x$ increments are progressively larger. Hence, in order to keep $\Delta y$, the logarithmic
increment, constant, it will be necessary to decrease the ordinate with each increase in the abscissa. This procedure suggests the hyperbolic curve, \( n = 1/x \), in an \((n,x)\) plane. Take the initial value of \( x \) as 1, since the logarithm of 1, and hence the area under consideration, is zero. The value of \( x \) increases by increments \( \Delta x_1, \Delta x_2, \ldots \Delta x_i \), until the final value of \( x \) is \( x_i \), the number whose logarithm we wish to determine. Hence the Bürgi logarithm of \( x_i \) is equal to the sum of all the rectangles so constructed (Fig. 3).

This representation leads to the natural logarithm, if, instead of letting \( \Delta y = \Delta x/x = 10^{-4} \), we consider \( \Delta x \) as approaching the limit \( dx \). Hence \( \Delta y \) approaches the limit \( dy \). The summation of all the infinitesimal rectangles between 1 and \( x_i \) will then represent the true area under the hyperbola. This procedure leads
to the integration of \(dy = \frac{dx}{x}\). When we consider the increment of the logarithm as approaching zero, we may replace Bürgi's equation
\[
x = (1 + \frac{1}{10^4})10^4 y
\]
by the equation
\[
x = (1 + \frac{1}{n})^n y
\]
where the value of \(n\) becomes infinite. But, by definition,
\[
\lim_{n \to \infty} (1 + \frac{1}{n})^n = e.
\]
Therefore, integrating \(dy = \frac{dx}{x}\) between the limits \(l\) and \(x_1\),
\[
y = \int_{l}^{x_1} \frac{dx}{x} = \left[ \log x \right]_{l}^{x_1} = \log x_1 - \log l = \log \frac{x_1}{l}
\]
thus verifying graphically the formulas for integrating \(\frac{dx}{x}\) and differentiating \(\log_e x\).

**Logarithmic Tables**

Logarithmic series is used to construct tables of logarithms. Since the logarithm of \(y^n\) is determined by the logarithm of \(y\), it is necessary to obtain only the logarithms of primes by means of the logarithmic series.

---

Meritz, Robert E. *Plane and Spherical Trigonometry*, p. 390
If \((1 + y) = e^k\), then \(k = \log_e(1 + y)\), and
\[
(1 + y)^x = e^{kx} = e^x \log_e(1 + y).
\]

Expanding the right hand member of (1), we find that \((1 + y)^x\) is equal to
\[
(2) \quad 1 + x \log_e(1 + y) + \frac{x \log_e(1 + y)^2}{2!} + \frac{x \log_e(1 + y)^3}{3!} + \cdots
\]

Expanding the left hand member of (1) by the binomial theorem, \((1 + y)^x\) is equal to
\[
(3) \quad 1 + xy + \frac{x(x - 1)y^2}{2!} + \frac{x(x - 1)(x - 2)y^3}{3!} + \cdots
\]

By Cauchy's ratio test, we find that (1) converges absolutely.\(^9\) Equation (2) also converges if \(y\) is less than 1.\(^10\)

Therefore, we may consider (2) and (3) as algebraic equations containing a finite number of terms. If we equate the two series and simplify by subtracting 1 from both sides and dividing by \(x\), we obtain
\[
\log_e(1 + y) \left[ 1 + x \log_e(1 + y) + \frac{x^2 \log_e^2(1 + y)}{2!} + \cdots \right]
= y + \frac{(x - 1)y^2}{2!} + \frac{(x - 1)(x - 2)y^3}{3!} + \cdots
\]

As the exponent, \(x\), approaches zero, the series on the

---

\(^9\)Woods, Frederick S. *Advanced Calculus*, p. 41.

\(^10\)Ibid., p. 38.
left in the bracket approaches 1, and the series on the right approaches 
\[ y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots, \]
hence, in the limit, 
\[ (4) \quad \log_e(1 + y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \ldots - \frac{(-1)^{n-1}y^n}{n} + \ldots \]
Similarly, if \( y \) is negative, but \( |y| < 1 \), we obtain 
\[ (5) \quad \log_e(1 - y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \frac{y^4}{4} - \ldots - \frac{(-1)^{n-1}y^n}{n} - \ldots \]
We may let \( y \) approach 1, and calculate the logarithm of \( 2 \) as nearly as we please by the series (4). However, (4) converges so slowly that, to obtain \( \log 2 \) accurately to four decimal places would require 100,000 terms.\(^{11}\)
In order to obviate this difficulty, (5) may be subtracted from (4), resulting in
\[
\log_e(1 + y) - \log_e(1 - y) = \log_e 1 + y = 2 \left[ y + \frac{y^3}{3} + \frac{y^5}{5} + \frac{y^7}{7} + \ldots \right] \frac{1}{1 - y}
\]
Substituting a new variable, \( 1/(2n + 1) \), for \( y \), we obtain 
\[
\log_e n + \frac{1}{n} = 2 \left[ \frac{1}{2n + 1} + \frac{1}{3(2n + 1)^3} + \frac{1}{5(2n + 1)^5} + \frac{1}{7(2n + 1)^7} + \ldots \right]
\]
or
\[ (6) \quad \log_e (n + 1) = \log_e n + 2 \left[ \frac{1}{2n + 1} + \frac{1}{3(2n + 1)^3} + \frac{1}{5(2n + 1)^5} + \frac{1}{7(2n + 1)^7} + \ldots \right] \]
This series is absolutely convergent if $y < 1$, or $n > 0$. If $\log_e n$ is known, for example, let $n = 1$, then $\log_e (n + 1)$, in this case, 2, may be calculated. From $\log_e 2$, $\log_e 3$ is determined, and hence a complete logarithmic table may be constructed. In practice, only the primes are treated by this series.

**Modulus of Common Logarithms**

To obtain common logarithms from natural logarithms, a modulus is determined. Let

$$e^x = N, \quad x = \log_e N,$$

$$10^y = N, \quad y = \log_{10} N.$$

Then,

$$e^x = 10^y = N, \text{ or } \frac{e^x}{y} = 10, \text{ and } \frac{x}{y} = \log_{10} 10 = 2.30259 \ldots$$

from which

$$x = 2.30259 y, \text{ or } \log_e N = 2.30259 \log_{10} N,$$

and

$$\log_{10} N = \log_e 10/2.30259 = 0.43429448 \log_e N.$$

The number 0.43429448 thus determined is called the modulus of the system of common logarithms. If we reverse this process to determine natural logarithms from common logarithms, the modulus is found to be 2.30259, which is also $\log_e 10$. 
A simple method of differentiating $e^x$ results from a consideration of the expansion of $e^x$ by the binomial theorem:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Taking the derivative of the right hand expression term by term,

$$\frac{d(e^x)}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \ldots$$

But this new series is identical with the original series. Hence

$$\frac{d(e^x)}{dx} = e^x.$$ Inversely,

$$\int e^x \, dx = e^x + C.$$
CHAPTER IV
THE LOGARITHMIC SPIRAL

The curve generated by a moving point which cuts obliquely all radii emanating from a common center at a constant angle is called a logarithmic, or equiangular, spiral. It describes an infinite number of circumvolutions about its pole, from which it constantly recedes.\(^1\)

Consider a logarithmic spiral with constant angle equal to 45°. Take that portion of the spiral cut off by one radian, and denote the smaller side of this angle by unity. Divide the angle into a large number, \(n\), of equal angles, each subtended by an infinitesimally small arc. We may consider the arcs as segments of straight lines.

Drop \(Aa, Bb, Cc\), (Fig. 4) perpendicular to \(OB, OC, OD\), respectively. Since the constant angle is 45°, the small triangles are isosceles, and \(Aa = aB, Bb = bC, Cc = cD\), etc. We may suppose \(OA = Oa, OB = Ob\),

\(^1\)De Bray, M. E. J. G. *Exponentials Made Easy*, p.3.
OC = Oc, etc., since 1/n is very small. Since length of arc is equal to the radius times the angle in radians, it follows that

\[ Aa = OA \times \frac{1}{n} = 1 \times \frac{1}{n} = \frac{1}{n}, \]
\[ Bb = OB \times \frac{1}{n}, \]
\[ Cc = OC \times \frac{1}{n}, \text{ etc.}, \]

and

\[ OA = 1 \]
\[ OB = Oa + AB = OA + AA = OA + OA \times \frac{1}{n} = OA(1 + \frac{1}{n}) = 1 + \frac{1}{n}, \]
\[ OC = OB + BC = OB + Bb = OB + OB \times \frac{1}{n} = OB(1 + \frac{1}{n}) = (1 + \frac{1}{n})^2, \]
\[ OD = OC + CD = OC + Cc = OC + OC \times \frac{1}{n} = OC(1 + \frac{1}{n}) = (1 + \frac{1}{n})^3, \]

and so on.

Comparing the length of the radius vector with the angle included between it and OA:

<table>
<thead>
<tr>
<th>Radius Vector</th>
<th>Angle θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0 radian</td>
</tr>
<tr>
<td>1 + 1/n</td>
<td>1/n</td>
</tr>
<tr>
<td>(1 + 1/n)^2</td>
<td>2/n</td>
</tr>
<tr>
<td>(1 + 1/n)^3</td>
<td>3/n</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>(1 + 1/n)^n</td>
<td>n/n = 1 radian</td>
</tr>
</tbody>
</table>
As \( N \) becomes infinite, the equation of the logarithmic spiral is

\[
\left(1 + \frac{1}{n}\right)^n = e
\]

\[
(1 + 1/n) = e^{1/n}
\]

\[
(1 + 1/n)^2 = e^{2/n}
\]

\[
(1 + 1/n)^3 = e^{3/n}, \text{ etc.}
\]

Substituting these values in the preceding table,

<table>
<thead>
<tr>
<th>Radius Vector</th>
<th>Angle ( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^0 = 1 )</td>
<td>0 radian</td>
</tr>
<tr>
<td>( e^{1/n} )</td>
<td>( 1/n )</td>
</tr>
<tr>
<td>( e^{2/n} )</td>
<td>( 2/n )</td>
</tr>
<tr>
<td>( e^{3/n} )</td>
<td>( 3/n )</td>
</tr>
<tr>
<td>[ \vdots ]</td>
<td>[ \vdots ]</td>
</tr>
<tr>
<td>( e^{n/n} = e )</td>
<td>( n/n = 1 ) radian</td>
</tr>
</tbody>
</table>

In each instance, the radius vector is equal to a power of \( e \) which is the value in radians of the angle \( \theta \) between it and the polar axis. Therefore, the equation of the logarithmic, or equiangular, spiral, whose constant angle is 45°, may be represented by the equation\(^2\)

\[
r = e^\theta
\]

Consider the spiral, \( r = e^\theta \), and the \( \theta \) spiral, \( r = 2.3026 \theta \). It is obvious that, in the \( r \) spiral, \( \theta = \log_e r \), and for the \( \theta \) spiral \( \theta = \log_{2.3026} r \). If an arc of a circle of radius \( r \) intersects each of these two arcs, the

\(^2\)De Bray, M. E. J. G. *Exponentials Made Easy*, p. 112.
The general form of the equation of the logarithmic spiral is

\[ r = ke^{m\theta}, \]

where \( k, a, \) and \( m, \) are constants. However, a number \( n \) may be found such that \( a^m = e^n, \) thus reducing the number of arbitrary constants to two in the equation

\[ r = ke^{n\theta}, \]

of the logarithmic spiral (Fig. 5). Here \( k \) determines the scale to which the spiral is drawn, and \( n \) depends on the angle at which the curve cuts the various radii vectors.

To obtain the common logarithmic spiral, the necessary and sufficient condition is that \( r = 10 \) when \( \theta = 1 \) in the equation (1). Then

\[ 10 = e^n, \]

\[ 1 = n \times \log_{10} e = 0.4343 \times n \]

\[ n = 2.3026, \]

and the equation of the common logarithmic spiral is

\[ r = e^{2.3026 \theta}. \]

Consider the \( e \) spiral, \( r = e^{\theta}, \) and the 10 spiral, \( r = e^{2.3026 \theta}. \) It is obvious that, in the \( e \) spiral, \( \theta = \log_e r, \) and for the 10 spiral, \( \theta = \log_{10} r. \) If an arc of a circle of radius \( r \) intersects each of these two
Fig. 5

The Logarithmic Spiral, $r = ke^{n\theta}$
spirals, the angle $\theta$ between the radius vector at the point of intersection and the polar axis is a measure of the logarithm of $r$ to the base $e$ in the first case and to the base 10 in the second case. In the 10 spiral, $r = e$ occurs at an angle of 0.4343 or 1/2.3026 radian, and in the e spiral, $r = 10$ occurs at 2.3026 radian. Hence $2.3026 = \log_{e}10$, $0.4343 = \log_{10}e$, and $\log_{e}10 = 1/\log_{10}e$.

Jacques Bernouilli discovered that the involute and the evolute of the logarithmic spiral are similar logarithmic spirals shifted to another position. The involute may be considered as the locus of a point on a string coiled on the spiral as the string is unwound.

Bernouilli's tomb is engraved with this device, together with the motto *Eadem mutata resurgo* (I rise again, changed but the same). 3

If the logarithmic spiral is caused to roll upon a straight path, its center will generate a straight line. 4

It is evident that, since the angle in radians between $r$ and the polar axis is the logarithm of $r$, we may utilize the curve for any logarithmic operation. Thus, to multiply two numbers, it is necessary only to add the angles determined by the radii vectors of lengths

3 Ibid., p. 5
4 Ibid., p. 120.
corresponding to the two numbers to obtain the logarithm of their product. The radius vector corresponding to this sum is therefore the product of the two numbers. This characteristic of the logarithmic spiral is made practicable by the construction of an extremely accurate transparent curve, manufactured commercially.  

Other uses of the curve are: (1) The curve gradually changing, it is peculiarly adapted for fitting to and drawing any part of a given curve; (2) it assists in the rapid determination of the center of curvature of a given curve, and (3) in drawing the evolute of a given curve.  

---

6 Ibid., p. 10.
CHAPTER V
THE CATENARY

The catenary is the curve assumed by a perfectly flexible chain of appreciable weight suspended at two points not in the same vertical line, and influenced by the force of gravity. The term catenary is derived from the Latin catenarius, chain-like.

We will derive the equation of the catenary and show its connection to the constant e.

Consider a portion AC (Fig. 6) of the chain, the point A being its lowest point. This portion is acted upon by three forces: its weight W acting at the center of gravity G; the tension T exerted by CD in resisting the weight of AC; the horizontal pull a exerted by the

---

Fig. 6

---

...
portion of the chain to the left of A. In the vector
triangle CMN,
\[ CM^2 = MN^2 + CN^2, \text{ or } T^2 = a^2 + w^2. \]

Represent AC by s, and an increment of s by ds. In Fig. 6, dx and dy represent horizontal and vertical increments, respectively. Since triangles abc and MCN are similar, and s = W,

\[ \frac{dx}{ds} = \frac{MN}{MC} = \frac{a}{T} = \frac{a}{\sqrt{(a^2 + w^2)}} = \frac{a}{\sqrt{(a^2 + s^2)}}. \]

Hence,
\[ dx = a ds/ \sqrt{(a^2 + s^2)} \]
and
\[ x = a \int ds/ \sqrt{(a^2 + s^2)}, \]
from which
\[ x = a \log_e \left[ s + \sqrt{(a^2 + s^2)} \right] + C. \]

Evaluating the constant of integration, when \( x = 0, s = 0 \). Then \( 0 = a \log_e (0 + a) + C \), and \( C = -a \log_e a \):
therefore,
\[ \frac{x}{a} = \log_e \left[ \frac{s + \sqrt{(a^2 + s^2)}}{a} \right] \]
and
\[ x/a = \log_e \left[ \frac{s + \sqrt{(a^2 + s^2)}}{a} \right] \]

Changing to exponential notation,
\[ s + \sqrt{(a^2 + s^2)} = ae^{x/a}. \]

---

1 Reynolds, Joseph B. *Analytic Mechanics*, pp. 210-211.
2 De Bray, op. cit., p. 151.
To obtain an expression involving $y$, a similar process is indicated.

$$\frac{dy}{ds} = \frac{CN}{CM} = \frac{W}{T} = \sqrt{(a^2 + W^2)} = \frac{s}{\sqrt{(a^2 + s^2)}},$$

from which

$$dy = s \frac{ds}{\sqrt{(a^2 + s^2)}} \quad \text{and} \quad y = \int s \frac{ds}{\sqrt{(a^2 + s^2)}}.$$

Hence

$$y = \sqrt{(a^2 + s^2)} + C.$$

Evaluating the constant of integration $C$, when $y = 0$, $s = 0$, so that

$$0 = \sqrt{(a^2 + 0)} + C,$$

and $C = -a$.

Hence

$$y + a = \sqrt{(a^2 + s^2)}.$$

Returning to the equation containing $x$,

(1) $$\sqrt{(a^2 + s^2)} + s = ae^{x/a},$$

and multiplying the left side by $\sqrt{(a^2 + s^2)} - s$,

$$\left[\sqrt{(a^2 + s^2)} + s\right]\left[\sqrt{(a^2 + s^2)} - s\right] = a^2 + s^2 - s^2 = a^2,$$

or

$$ae^{x/a} \left[\sqrt{(a^2 + s^2)} - s\right] = a^2,$$

from which

(2) $$\sqrt{(a^2 + s^2)} - s = ae^{-x/a}.$$
Adding (1) to (2),

\[ \sqrt{a^2 + s^2} = a(e^{x/a} + e^{-x/a}) \]

or

\[ y + a = \frac{a}{\xi}(e^{x/a} + e^{-x/a}) \]

Translating the x axis to a distance \( a \) below \( AX \),
y + \( a \) becomes the new value of \( y \), and the equation of the catenary becomes, in general

\[ y = \frac{a}{\xi}(e^{x/a} + e^{-x/a}) \]

A development of the equation of the catenary which is much more concise, but essentially the same in method of attack, arises from a consideration of hyperbolic functions.

---

3 McMahon, James. *Hyperbolic Functions*, p. 47.
Hyperbolic functions may be defined from either of two viewpoints. A purely analytic definition arises from a consideration of power series, and a geometric interpretation results from a graphical study of the hyperbola.

Analytical Definition

Consider the three power series resulting from Maclaurin's expansion of \( e^x \), \( \sin x \), and \( \cos x \):

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots,
\]

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \cdots,
\]

\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^k \frac{x^{2k}}{(2k)!} + \cdots.
\]

In the expansion of \( e^x \), replace \( x \) by \( xi \), where \( i = \sqrt{-1}, i^2 = -1, i^3 = -i, i^4 = 1, \) etc. Then

\[
e^{xi} = (1 - x^2/2! + x^4/4! - \cdots) + i(x - x^3/3! + x^5/5! - \cdots)
\]

\[= \cos x + i \sin x.\]

Expanding \( e^{-xi} \),

\[
e^{-xi} = \cos x - i \sin x.
\]

---

1 Woods, Frederick S. *Advanced Calculus*, pp. 53-56.
Solving (1) and (2) for \( \sin x \) and \( \cos x \),

\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i},
\]

\[
\cos x = \frac{e^{ix} + e^{-ix}}{2}.
\]

Equations (3) and (4) show that any trigonometric function may be expressed in terms of the imaginary and the constant \( e \).

Since

\[
(e^{\theta i})^n = e^{n\theta i},
\]

DeMoivre's theorem follows from (1):

\[
(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta
\]

The hyperbolic functions are defined by analogy from (3) and (4) thus:

\[
\sinh x = \frac{e^x - e^{-x}}{2},
\]

\[
\cosh x = \frac{e^x + e^{-x}}{2},
\]

\[
\tanh x = \frac{\sinh x}{\cosh x},
\]

\[
\coth x = \frac{1}{\tanh x},
\]

\[
\text{sech} x = \frac{1}{\cosh x},
\]

\[
\text{cosech} x = \frac{1}{\sinh x}.
\]

Replacing \( x \) by \( ix \) in (3) and (4),
the equalities referring to sign as well as to magnitude, then \( P_1, P_2 \) are are \( \sin ix = i \sinh x, \cos ix = \cosh x. \) Maclaurin's expansion for \( \sinh x \) and \( \cosh x \) are, from \( (5) \) and \( (6) \),

\[
(7) \quad \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2k-1}}{(2k-1)!} + \ldots
\]

\[
(8) \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2k}}{(2k)!} + \ldots
\]

Differentiating \( (7) \) and \( (8) \) term by term,

\[
\frac{d}{dx} (\sinh x) = \cosh x,
\]

\[
\frac{d}{dx} (\cosh x) = \sinh x.
\]

**Geometrical Definition**

To develop hyperbolic functions geometrically, it is necessary to formulate a few preliminary definitions. Consider a point \( P_1 \) on any central conic, and another point \( P_2 \) on a central conic of the same species. Let \( a_1, a_2 \), and \( b_1, b_2 \), be conjugate radii on the two conics respectively, and consider the \( a \)-radii as abscissae and the \( b \)-radii as ordinates. Now if the points \( P_1, P_2 \), be so situated that

\[
\frac{x_1}{a_1} = \frac{x_2}{a_2}, \quad \frac{y_1}{b_1} = \frac{y_2}{b_2}
\]

the equalities referring to sign as well as to magnitude, then \( P_1, P_2 \), are called corresponding points in the two systems. If \( Q_1, Q_2 \), be another pair of correspondents, then the sector and triangle \( P_1 Q_1 \) are said to correspond with the sector and triangle \( P_2 Q_2 \).

The "measure" of any area connected with a given central conic is the ratio which it bears to the constant area of the triangle formed by two conjugate diameters of the same conic. Thus, the measure of the sector \( A_0 P_1 \) is the ratio \( \frac{\text{sector } A_0 P_1}{\text{triangle } A_0 B_1} \), regarded as positive or negative according as \( A_0 P_1 \) and \( A_0 B_1 \) are at the same or opposite sides of their common initial line (Fig. 7, page 39).

It may be shown that the areas of corresponding triangles and the areas of corresponding sectors have equal measures.\(^3\) Thus, where the measuring triangles \( A_1 B_1, A_2 B_2 \), are denoted by \( K_1, K_2 \), the corresponding triangles by \( T_1, T_2 \), and the corresponding sectors by \( S_1, S_2 \), we have

\[
\frac{T_1}{K_1} = \frac{T_2}{K_2}, \quad \frac{S_1}{K_1} = \frac{S_2}{K_2}.
\]

The ratios \( x/a, y/b \), are called characteristic ratios of the given sectorial measure \( S/K \), and are constant in magnitude and sign for all sectors of the same measure and species wherever situated. Therefore, there

\(^3\) Ibid., p. 9, Arts. 2 and 3.
exists a functional relation between the sectorial measure and each of its characteristic ratios.

With these fundamental definitions, we may proceed with the development of the definition of the hyperbolic function. Let \( P_1, P_2 \) be corresponding points on an ellipse and a circle, referred to the conjugate axes \( O_1A_1, O_1B_1 \), and \( O_2A_2, O_2B_2 \) (Fig. 7).

\[ \text{Fig. 7} \]

Let the angle \( \angle AOP = \theta \) in radians. Then

\[ \frac{S_2}{K_2} = \frac{1}{2}(a_2)^2 \theta = \theta, \]

from which

\[ x_2/a_2 = \cos S_2/K_2, \quad y_2/b_2 = \sin S_2/K_2, \quad (a_2 = b_2). \]

Hence, in the ellipse, since \( P_1 \) and \( P_2 \) are corresponding points,

\[ x_1/a_1 = \cos S_1/K_1, \quad y_1/b_1 = \sin S_1/K_1. \]
Since the ellipse and the hyperbola are not of the same species, we cannot conclude that the above relation holds for the hyperbola. But, since there exists a functional relationship between the characteristic ratios and the sectorial measure of any given sector in the hyperbola, we may set up the following relation by analogy with the above procedure:

\[ \frac{x_1}{a_1} = \cosh \frac{S_1}{K_1}, \quad \frac{y_1}{b_1} = \sinh \frac{S_1}{K_1}. \]

These equations are definitions of the hyperbolic cosine and hyperbolic sine, and express that the ratio of the two lines on the left is a certain definite function of the ratio of the two areas on the right.

For brevity, we may write \( \frac{S_1}{K_1} = u \). The tangent, cotangent, secant, and cosecant are defined for the hyperbolic functions in a manner analogous to the corresponding trigonometric functions. From the equation of the hyperbola,

\[ \frac{(x_1)^2}{(a_1)^2} - \frac{(y_1)^2}{(b_1)^2} = 1, \]

we may write

\[ \cosh^2 u - \sinh^2 u = 1. \]

To show that the geometrical definition is identical to the analytical definition, we will proceed with a
discussion of the variations of the hyperbolic functions from the geometric standpoint and expand \( \sinh u \) and \( \cosh u \) by Maclaurin's series, arriving at the values determined in (5) and (6).

The value of \( u \) depends only upon the sectorial measure \( S \), since in the definition \( u = S/K \), (neglecting the subscripts), \( K \) remains constant. Consider the sectors of one-half of a rectangular hyperbola whose conjugate radii are equal, and take the principal axis \( 0a = a \) as the common initial line of all sectors (Fig. 8).

It is evident that as \( P \) comes from infinity on the lower branch, passes through \( A \), and to infinity on the upper branch, the sectorial measure passes from negative infinity through zero to positive infinity, assuming every intermediate value. Since the functions \( \sinh u \) and \( \cosh u \), for any position of \( OP \) are equal to the ratios of \( x \) and \( y \) to the principal radius \( a \),

\[
\cosh 0 = 1, \quad \sinh 0 = 0, \quad \tanh 0 = 0.
\]
It may be shown from geometric considerations that
the derivative of \( \sinh u = \cosh u \), and the derivative
of \( \cosh u = \sinh u \).

Expanding \( \sinh u \) by Maclaurin's series, useful
in every branch of physics and in the applications
of physics, whether to observational data or to
bulk properties, such as a nuclear, mechanical,
and since
\( f(0) = \sinh 0 \), \( f'(0) = \cosh 0 \), \( f''(0) = \sinh 0 \), ..., 
and, as shown above, \( \sinh 0 = 0 \) and \( \cosh 0 = 1 \), then

\[
\sinh u = u + \frac{1}{3!} u^3 + \frac{1}{5!} u^5 + \ldots.
\]

Similarly,

\[
\cosh u = 1 + \frac{1}{2!} u^2 + \frac{1}{4!} u^4 + \ldots.
\]

By comparing (9) and (10) with (7) and (8), it is
seen that the expressions are identical, thus establishing
the identity of the analytical and geometrical definitions
of the hyperbolic functions.

**Applications of Hyperbolic Functions**

C. D. Walcott, former secretary of the Smithsonian
Institution, gives the following statement on the value
and applications of the hyperbolic functions.\footnote{Moritz, Robert E. \textit{Plane and Spherical Trigonometry}, p. 353.}

Hyperbolic functions are extremely useful in every branch of physics and in the applications of physics, whether to observational and experimental sciences or to technology. Thus, whenever an entity (such as light, velocity, electricity or radioactivity) is subject to gradual extinction or absorption, the decay is represented by some form of Hyperbolic Functions. Whenever mechanical strains are regarded as great enough to be measured they are most simply expressed in terms of Hyperbolic Functions. Hence geological deformations invariably lead to such expressions.

\footnote{Moritz, Robert E. \textit{Plane and Spherical Trigonometry}, p. 353.}
CHAPTER VII
THE TRANSCENDENTALISM OF e

In 1844 Joseph Liouville (1809-1882) read before the Paris Academy a paper under the title, "On the very extensive class of quantities which are neither algebraic nor even reducible to algebraic irrationals." Fifty years later Georg Cantor (1845-1918) placed the theory of these non-algebraic numbers, called transcendental, on a solid foundation. Their existence was suspected as early as 1794 by Legendre (1752-1833).\(^1\)

The transcendental equation \(x^i\pi + 1 = 0\) was shown by Euler (1707-1783) to have \(e\) for one of its roots. Felix Klein states that this equation "is certainly one of the most remarkable in all mathematics."\(^2\)

Charles Hermite (1822-1901) proved in 1873, by means of a unique function known as "Hermite's integral", that \(e\) is a transcendental number. From the relation given in the equation \(x^i\pi + 1 = 0\), Lindemann proved nine years later that \(\pi\) is also a transcendental.\(^3\)

The following proof of the transcendentalism of \(e\) is based upon a simplification of Hermite's method,

\[^1\]Dantzig, Tobias. *Number, the Language of Science*, p. 112.

\[^2\]Miller, G. A. *Historical Introduction to Mathematical Literature*, p. 141.

given by Hilbert in volume 43 of the Mathematische Annalen (1893).\(^4\) Hurwitz and Gordan also modified Hermite's proof.\(^5\)

The proof lies in demonstrating that if it is assumed that \(e\) satisfies the equation

\[
(1) \quad a_0 + a_1 e + a_2 e^2 + \ldots + a_n e^n = 0,
\]

where \(a_0 \neq 0\), and \(a_0 \ldots a_n\) are integers, then the assumption leads to a contradiction. Therefore, the assumption is false, and \(e\) satisfies no algebraic equation, and hence is transcendental.\(^6\)

We will approximate \(e\) and powers of \(e\) by means of rational numbers, obtaining

\[
(2) \quad e = \frac{M_1 + \varepsilon_1}{M}, \quad e^2 = \frac{M_2 + \varepsilon_2}{M}, \ldots, \quad e^n = \frac{M_n + \varepsilon_n}{M},
\]

where \(M, M_1, M_2, \ldots, M_n\) are integers, and \(\varepsilon_1/M, \varepsilon_2/M, \ldots, \varepsilon_n/M\) are very small positive fractions. Multiplying (1) by \(M\) and rearranging,

\[
(3) \quad \left[ \frac{a_0 M + a_1 M_1 + a_2 M_2 + \ldots + a_n M_n}{M} \right] + \left[ \frac{a_1 \varepsilon_1 + a_2 \varepsilon_2 + \ldots + a_n \varepsilon_n}{M} \right] = 0.
\]

The first parenthesis may be proved to be an integer not equal to zero. In the second parenthesis, the values of \(\varepsilon_1\) may be made so small that the sum of the terms is a

---

\(^4\) Klein, Felix. Elementary Mathematics from an Advanced Standpoint, p. 238.

\(^5\) Hobson, op. cit., p. 306.

\(^6\) Klein, op. cit., p. 238.
proper fraction. Then the sum of an integer and a proper fraction will be zero, which is obviously impossible.

We shall define M by means of Hermite's integral,

\[ M = \int_{0}^{\infty} z^{p-1} \frac{[ (z-1)(z-2) \cdots (z-n) ]^p e^{-z}}{(p-1)!} \, dz \]

in which \( n \) is the degree of the assumed equation and \( p \) is an odd prime to be determined. Multiplying (2) by \( M \), we have for any particular power of \( e \), \( e^i \),

\[ M e^i = M_1 + \epsilon_1. \]

By breaking the interval of integration at \( i \),

\[ M_1 = e^i \int_{0}^{i} z^{p-1} \frac{[ (z-1)(z-2) \cdots (z-n) ]^p e^{-z}}{(p-1)!} \, dz, \]

and

\[ \epsilon_1 = e^i \int_{0}^{i} z^{p-1} \frac{[ (z-1)(z-2) \cdots (z-n) ]^p e^{-z}}{(p-1)!} \, dz. \]

Expanding \( [ (z-1)(z-2) \cdots (z-n) ]^p \) in (4) by the polynomial theorem,

\[ [ (z-1)(z-2) \cdots (z-n) ]^p = z^n p + \cdots + (-1)^n n! (n \cdot n! \cdots n!)^p, \]

from which the Hermite integral becomes
\[(7) \quad M = \frac{(-1)^n(n!)^p}{(p - 1)!} \int_0^\infty z^{p-1} e^{-z} dz + \sum_{q=p+1}^{np+p} \frac{C_q}{(p-1)!} \int_0^\infty z^{q-1} e^{-z} dz,\]

where \(C_q\) are integral constants. The integrals in (7) are Gamma functions; the value of the first is \((p - 1)!\), and the value of the second is \((q - 1)!\). Hence, (7) becomes

\[M = \frac{(-1)^n(n!)^p}{(p - 1)!} + \sum_{q=p+1}^{np+p} \frac{C_q}{(p-1)!} \frac{(q - 1)!}{(p - 1)!},\]

Since \(q\) is always larger than \(p\), \((q - 1)!/(p - 1)!\) is an integer, and contains the factor \(p\). Taking \(p\) out of the sum,

\[M = \frac{(-1)^n(n!)^p}{(p - 1)!} + p \left[ \frac{C_{p+1}}{p+1} + \frac{C_{p+2}}{p+2} + \frac{C_{p+3}}{(p+1)(p+2)} + \ldots \right].\]

If \(M\) is divisible by \(p\), then the first term, \((-1)^n(n!)^p\), must be divisible by \(p\). But \(p\) will not be a divisor of this term if it is not a divisor of one of its factors \(1, 2, \ldots, n\). Since \(p\) is prime, this condition will not be met if \(p\) is taken larger than \(n\). Consequently, it is possible to so select \(p\) that \(M\) is not divisible by \(p\).

In (3), \(a_0 \neq 0\). If \(p\) is taken larger than \(|a_0|\), and at the same time so that \(M\) is not divisible by \(p\), then the product \(a_0^M\) which is the first term of (3) is not divisible by \(p\).
If the remaining terms of (3) are divisible by $p$, then (3) is not divisible by $p$. We will examine the coefficients $M_i$ for divisibility by $p$. In (5), placing the factor $e^i$ under the integral sign and introducing a new variable of integration $x = z - i$, which changes the limits from $(i, \infty)$ to $(0, \infty)$, we have

\begin{equation}
M_i = \int_0^\infty \frac{(x + i)^{P-1} \left[(x + i-1)(x + i-2)\ldots(x + i-n)\right] e^{-x} dx}{(p - 1)!}
\end{equation}

In multiplying the factors in the numerator, the lowest power of $x$ is found to be $x^P$, and every term will contain this or some higher power of $x$. In determining the highest power of $x$, we observe that there are $n$ factors in the bracket. Hence, the highest power in the bracket is $n$, which, multiplied by the exponent $p$ and added to the highest power of $x$ in the parenthesis, $p - 1$, results in $np + p - 1$, or $(n + 1)p - 1$. Hence the integral of the numerator will be the sum of the integrals

\begin{align*}
&\int_0^\infty x^P e^{-x} dx, \\
&\int_0^\infty x^{P+1} e^{-x} dx, \\
&\int_0^\infty x^{(n+1)p-1} e^{-x} dx,
\end{align*}

together with their constant coefficients. Each of these integrals is a Gamma function whose value is $p!$, $(p + 1)!$, ..., $[(n + 1)p - 1]!$. Therefore, the numerator will be a summation of terms each of which contains the factor $p!$. Factoring $p!$ from this sum, there results $p!$ multiplied by
a whole number $A$, from which

$$M_i = \frac{p! A_i}{(p - 1)!} = p A_i, \quad (i = 1, 2, \ldots, n).$$

Therefore, every value of $M$ is a whole number which is divisible by $p$, and since the first term of the left parenthesis of (3) is not divisible by $p$, it follows that the left parenthesis of (3) is not divisible by $p$, and is therefore different from zero, for zero is divisible by $p$.

We will next consider the parenthesis on the right in (3) and show that each $\varepsilon_i$ may be made arbitrarily small by the selection of a sufficiently large value of $p$.

Let us examine the integrand of (6), with the value $e^1$ under the sign of integration,

$$\varepsilon_i = \int_0^z \frac{z^{p-1} (z - 1)(z - 2) \ldots (z - n) p e^{-z+1}}{(p - 1)!} \, dz$$

for values of $i$ between 0 and $n$. Setting the integrand equal to $y$, we see that when $z = 1, 2, 3, \ldots, n$, the ordinate will be zero, since the portion of the integrand $(z - 1)(z - 2) \ldots (z - n)$ will contain zero as a factor. Also, $dy/dz = 0$, indicating that the curve is tangent to the $z$-axis at these points (Fig. 9, p. 50).

If $z$ is $\frac{1}{2}$, then all the terms in $(z - 1)(z - 2) \ldots (z - n)$ will be negative. If $z$ is $\frac{1}{2}$, all the terms except the first will be negative, thus changing the sign of the
product. This change of sign continues for all fractional values of \( z \) between the integral values 1, 2, ..., \( n \). Hence the sign of the factor \((z - 1)(z - 2)\ldots(z - n)^p\) depends on \( p \), and since \( p \) is odd, the curve fluctuates above and below the \( z \)-axis. Moreover, if \( p \) is large, the denominator is large, and hence the function remains near the \( z \)-axis.

Fig. 9

Rearranging the numerator of the integrand of (6),

\[
z^{p-1}[(z-1)(z-2)\ldots(z-n)^p]e^{-z+i} \, dz
\]

has now become

\[
z^{p-1}[(z-1)(z-2)\ldots(z-n)]e^{-z+i} \, dz
\]

Assume \( G \) to be the maximum of the absolute value of \( z(z - 1)(z - 2)\ldots(z - n) \), and \( g_1 \) to be the maximum of the absolute value of \( (z - 1)(z - 2)\ldots(z - n)e^{-z+i} \), in the interval \((0, n)\). Then

\[
|\mathcal{E}_1| \leq \left[ \int_0^i \frac{G^{p-1}g_1 \, dz}{(p - 1)!} \right] = \frac{G^{p-1}g_1 \cdot i}{(p - 1)!},
\]
G, g₁, and i are independent of p. The denominator, (p - 1)!, will, for sufficiently large p, increase faster than Gᵖ⁻¹. Hence, the fraction Gᵖ⁻¹/(p - 1)!
becomes infinitesimally small with an infinitely large p.

It is evident, then, that with every value of ε₁, ε₂, sufficiently small, the entire right hand bracket of (3) becomes and remains less than any chosen number, and, in particular, less than unity. This condition, the condition that M is not divisible by p, and the condition that a₀ is not divisible by p, all depend on a sufficiently large p. Therefore, there must exist a value of p which will satisfy all three conditions.

The left member of equation (3),

\[
\left[ a₀M + a₁M₁ + \cdots + aₙMₙ \right] + \left[ a₁ε₁ + a₂ε₂ + \cdots + aₙεₙ \right] = 0
\]

has now been proven to consist of a non-vanishing integer increased by a proper fraction. It is obvious that this sum cannot equal zero. Hence, the assumed equation, (1),

\[
a₀ + a₁e¹ + a₂e² + \cdots + aₙeⁿ = 0,
\]
cannot exist, proving the transcendence of e.
CHAPTER VIII
APPLICATIONS OF e

Properties of e

The principal applications of e are based upon the following distinct properties of e:

1. It is the base of the natural system of logarithms.

2. It appears in equations of special curves, most applicable of which is the catenary.

3. Whenever the increase in a quantity is proportional to the magnitude of the quantity, the law of exponential growth, the statement of which involves e, is applied.

4. Since the derivative of $e^x$ is $e^x$, functions in which e appears are used (a) as integrating factors in certain differential equations, (b) in the Gamma function, and (c) in the equation of the probability curve.

As was shown in Chapter III, since the logarithm of 1 to the base e is known from the definition of a logarithm, the logarithm of 2 can be calculated. From the logarithm of 2, the logarithm of 3 may be calculated,
and so on. From the table of natural logarithms, the common logarithms are determined with the aid of the modulus 0.43429448\ldots. Thus e is the basis of all logarithmic calculation.

The Catenary

Since, by definition, when any flexible, inextensible solid is suspended from two points not in the same vertical line, the curve described is, approximately, a catenary, the equation of the catenary is important in the solution of problems arising in the construction of suspension bridges, high tension lines, cables, etc. Two examples of such problems are here stated.

(1) A power company wishes to suspend a high tension cable carrying 100,000 volts of electricity across the right of way of a railroad company on a horizontal span of 740 ft. with one point of support 53 ft. below the other. The roadbed is 30 ft. below the lower support and requires a clearance of 20 ft. below the lowest point of the cable. If copper wire 0.57 in. in diameter is being used, what factor of safety can be assured the railroad company?\footnote{Reynolds, Joseph B. Analytic Mechanics, p. 217.}

(2) Find the length of each cable, and find the maximum tension for a span of a suspension bridge 1000 ft.
long, the maximum deflection being 100 ft. and the weight of the span 4 tons per ft., there being two cables.²

The Law of Exponential Growth

The law of exponential growth is most easily comprehended from a study of compound interest. Consider \( C_0 \) dollars compounded at \( k \) rate annually, where \( k \) is a proper fraction. Then, at the end of \( x \) years, the capital \( C \) is represented by the equation³

\[
C = C_0(1 + k)^x.
\]

Now let \( C_0 \) be compounded \( n \) times annually. Then

\[
C = C_0(1 + k/n)^{nx}.
\]

Let \( k/n = 1/N \); then \( n = Nk \), and

\[
C = C_0(1 + 1/N)^{Nkx}.
\]

As \( n \) becomes infinite, \( N \) also becomes infinite, and

\[
\lim_{N \to \infty} C_0(1 + 1/N)^{Nkx} = C_0e^{kx}.
\]

Consequently, when the change in a quantity which is continuously changing is proportional to the magnitude of the quantity, then⁴

\[
C = C_0e^{kx}.
\]

²Ibid., p. 221.
³Daniels, Farrington. Mathematical Preparation for Physical Chemistry, p. 128.
⁴De Bray, M. E. J. G. Exponentials Made Easy, p. 87.
is, in general, a constant of proportionality \( k \), whose value when \( x \) is zero is \( C_0 \), or the initial value of \( C \). The symbol \( k \) is the rate of change, and may be either positive or negative.

The phenomenon of exponential growth occurs frequently in chemistry and physics. Below are given equations of examples of this nature.

(1) **Condenser Discharge.** The quantity of charge \( q \) remaining after \( t \) seconds is expressed by the equation

\[
q = Qe^{-t/RC}
\]

where \( Q \) is the initial charge in coulombs, \( C \) is the capacitance in farads, and \( R \) is the resistance of the circuit in ohms.

(2) **Atmospheric Pressure.** Atmospheric pressure, \( p \), decreases at a rate proportional to the altitude, determined by the equation

\[
p = p_0e^{-kh}
\]

in which \( p_0 \) is the pressure at sea level, and \( h \) represents the altitude.

(3) **Concentration of Reacting Materials.** Where \( c_0 \)

---

5 Shepardson, George D. *Elements of Electrical Engineering*, p. 152.
7 Ibid., p. 138.
is the initial concentration, the concentration $c$ at time $t$ is given by the equation

$$C = c_0 e^{-kt}.$$  

(4) **Ionization by Collision.** Where $n_0$ is the number of ions generated per cubic cm. of gas by the ionizing agent at a distance $x$ from the positive electrode, and $\alpha$ is the number of ions which a single negative ion produces by collision in 1 cm. of its path, $n$ the total number of negative ions present per unit volume at the point considered, then $n$ is represented by the equation

$$n = n_0 e^{\alpha x}.$$  

(5) **Absorption of X-rays.** Where $I$ is the ionization current, which gives a measure of the intensity of the x-rays, $t$ a very small thickness of the absorbing material, and $\mu$ is a constant of proportionality, the ionization current $I$, and hence the intensity of the x-rays, after absorption, is given by the equation

$$I = I_0 e^{-\mu t},$$  

where $I_0$ is the intensity without the absorbing sheet.

**Applications of the derivative of $e^x$.**

In the general type of linear differential equation

8 Crowther, J. A. Ions, Electrons, and Ionizing Radiations, p. 60.

of the first order

\[ \frac{dy}{dx} + Py = Q, \]

where \( P \) and \( Q \) are functions of \( x \) only, an integrating factor is found to be \( \int P \, dx \), since

\[ \int P \, dx = \int P \, dx \quad \frac{d}{dx} \left( ye^{-P} \right) = e^{(dy/dx + Py)}. \]

Introducing this factor, the solution of the differential equation becomes

\[ ye^{-P} = \int Q e^{P} \, dx + C. \]

The Gamma function of \( n \) is defined by the equation\(^{11}\)

\[ \Gamma(n) = \int_{0}^{\infty} x^{n-1} e^{-x}, \]

where \( n \) is greater than zero, and is equivalent to \((n - 1)!\); and, in conjunction with the beta function, is necessary in the solution of complicated definite integrals.

The probability curve is satisfied by an equation

\[^{10}\text{Cohen, Abraham.} \text{ An Elementary Treatise on Differential Equations, 1906, p. 18.}\]

\[^{11}\text{Woods, Frederick S.} \text{ Advanced Calculus, p. 164.}\]
of the type

\[ y = a^{-x^2} \]

where \( x \) and \( y \) are coordinates, and \( a \) is any constant. The negative sign is imperative, since for large values of \( x, y \) must be small. The exponent 2 is necessary, since the curve is symmetrical with respect to the y axis. The constant, \( a \), may have any value, but \( e \) is universally adopted because of its simple differentiation and integration. Hence, the general equation of the probability curve is

\[ y = e^{-x^2}. \]

---

BIBLIOGRAPHY


A standard historical reference.


A brief history from an educational viewpoint.


Includes the use of e as an integrating factor.


An account of the theory and use of the logarithmic spiral.


A reliable source of the trends of electronic theory up to 1929.
Includes an excellent chapter on adaptations of e.

A treatise on the cultural value of mathematics.

A semi-popular compilation of miscellaneous data concerning exponential and logarithmic functions. A valuable reference.

Includes Gordan's proof of the transcendence of e.

A summary of Napier's work, honoring the tercentenary of the invention of logarithms.

A standard text on elementary calculus.

Includes applications of e.

Includes a lucid account of the historical development of the theory of logarithms, and a detailed proof of the transcendence of e according to Hilbert. A valuable reference.

Includes a geometrical development of hyperbolic functions.

An excellent survey of the literature of mathematics.


White, W. F.  Scrapbook of mathematics.  2nd ed.  
248p.  
  Includes an account of Boorman's calculation of e.

Woods, Frederick S.  Advanced calculus.  Boston,  
  Includes an analytic development of hyperbolic functions.