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Methods of Obtaining Asymptotes

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Methods of Obtaining Asymptotes.

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fulfillment of the requirements
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Degree.

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Introduction

The purpose of this thesis is to present in logical order the more important methods of obtaining asymptotes. In several cases where two or more different methods for obtaining the same type of asymptotes were found the simpler method is given first and then the more difficult method. Most of the simpler methods are based on Analytic Geometry, while the more difficult are based on the Calculus.

The first difficulty encountered was that of definition. It seemed that nearly every mathematician gave a different definition and based his discussion on it. The author has tried to base his discussion on two definitions which were thought to be the best.

A brief history is given to show, in a way, the importance of asymptotes and to give the names of the mathematicians who have done the larger amount of work in this field. This subject has interested mathematicians for many years, and is still receiving a great deal of attention.

Examples are given for all the methods discussed, also figures for most of the examples. The figures have been placed together at the end of the thesis. It was thought that this arrangement would give a much neater

appearance to the thesis, and, as the figures are numbered, nothing would be lost.

Any discussion on asymptotic lines as used in Projective Geometry has been omitted as it requires a knowledge of advanced mathematics which the author does not possess.

"Differential Calculus" by Joseph Edwards and "Curve Tracing" by Percival Frost proved to be the most helpful references. Acknowledgment is also due Prof. E. E. Colyer of the Fort Hays Kansas State College for the many helpful suggestions he has given in the preparation of this thesis.

I. Definitions.

The word "asymptote" is derived from three Greek words, ἀ, not; σὺν, together; πιπτω, fall; and it means "not falling together" or "coinciding".(i) The Greek definition for asymptote comes from the meaning of the word and is: "Some lines exist which approach indefinitely and yet remain not together falling".(ii)

There are many modern definitions and the following are a few of the more widely used.

"A line which approaches continually nearer to a given curve but does not meet it within a finite distance of the origin".(iii)

"A tangent to a curve at a point infinitely distant from the origin".(iv)

"A line which the curve approaches as one of the variables approaches infinity is called an asymptote to the curve".(v)

"An asymptote of a curve is a line towards which the curve finally approaches as it recedes from the origin to an infinite distance and from which the distance

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(i) McGiffert, Pl. & Solid Ana. Geom., p.171; Davies & Peck, Dict. of Math., p.52; Candy, Pl. & Solid Ana. Geom., p.162.

(ii) Ency. Brit., vol.2, p.594.

(iii) ibid., p.594; Gale & Watkeys, Elementary Functions, p.25.

(iv) Ency. Brit., vol.2, p.594; Frost, Curve Tracing, p.5; Love, Ana. Geom., p.200.

(v) Mason & Hazard, Ana. Geom., p.50; Woods & Bailey, A Course in Math., p.128.

of the points on the curve becomes less than any assignable quantity".(i)

"If a straight line cut a curve in two points at an infinite distance from the origin and yet is not itself wholly at infinity, it is called an asymptote to the curve".(ii)

"An asymptote to a plane curve is a straight line, lying partly within the finite region, which is the limiting position of a tangent to the curve as the point of tangency recedes indefinitely along an infinite branch of the curve".(iii)

These definitions can be divided into two groups. The first group includes the definitions which state that an asymptote is a tangent to a curve at infinity. The second group includes those which define an asymptote as a line which the curve approaches but does not meet within a finite distance of the origin. It is very easy to see that the difference between the two groups is principally in the region in which they are defined. The first group being defined in the infinite region and the second group in the finite region. Many authors give two

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(i) Edwards, Diff. Calculus, p.193; Frost, Curve Tracing, p.5.

(ii) Edwards, Diff. Calculus, p.182.

(iii) ibid., p.193; Harding & Mullins, Ana. Geom., p.195; Love, Ana. Geom., p.200; Carmichael & Weaver, The Calculus, p.231; Nichols, Ana. Geom., p.162; Candy, Pl. & Solid Ana. Geom., p.162; Murray, Infin. Calculus, p.286; Granville, Diff. & Int. Calculus, p.249.

definitions, one from each group, showing that this difference is of small importance and that either one is equally acceptable. Most of the proofs which are to follow are based on the last definition given on the preceeding page so it is the one which will be used in most of this discussion.

II. Brief History.

The Greek term for asymptotes occurs first in the writings of Apollonius (approximately 250-200 B. C.)(i). He was educated in Alexandria under the successors of Euclid and became known as "The Great Geometer". He wrote several books among which was one on asymptotes, axes, and diameters.

Menaechmus (2nd or 3rd century B. C.)(ii) might have determined the asymptotes of hyperbolas but as the original copies of his writings were lost there is no definite proof of this.

Geminus (1st century B. C.)(iii) gave a definition for asymptotes and discussed the asymptotes of the hyperbola and conchoid.

The next known writer on the subject was Franciscus Maurolycus (1493-1575)(iv) of Messina, known as

(i) Cajori, Hist. of Math., p.40; Smith, Hist. of Math. Special Topics of Elem. Math., p.318.
(ii) Allman, Greek Geom. From Thales to Euclid, p.169.
(iii) Ency. Brit., vol. 2, p.594.
(iv) Cajori, Hist. of Math., p.142.

the greatest geometer of his time. He discussed asymptotes more fully than Apollonius and applied them to various astronomical and physical problems.

John Wallis (1616-1703)(i) showed that the space between a curve and its asymptote was infinite.

Among the earlier writers, Jean Paul de Gua (1713-1785)(ii) did the most work on asymptotes. He showed how to find asymptotes, tangents, and various singular points of curves of all degrees and proved, by perspective, that several of these points can be at infinity.

Some of the men of more modern times who have worked on asymptotes are; David Hilbert of Gottingen, Maxine Bocher of Harvard, Max Mason of the University of Wisconsin, Mauro Picone of Turin, R. M. E. Mises of Stratsbury, H. Weyl of Gottingen, George D. Birkhoff of Harvard.(iii) The writer was unable to determine the exact contribution of each of these mathematicians but they are known for the excellence of their work.

III. Comparison of Asymptotes and Other Curves Which Resemble Them.

It is readily seen from the definitions of

(i) Cajori, Hist. of Math., p.185.
(ii) ibid., p.224.
(iii) ibid., p.391.

asymptotes that they belong to the class of curves which include tangents, envelopes, evolutes, and involutes. These lines or curves all help to determine the original curve and might well be called "auxillary lines".

The tangent line is generally defined(i) as the limiting position of the secant as the points of intersection of the secant with the curve approach coincidence, the point of coincidence being called the point of tangency. From some of the definitions of asymptotes we see at once that they are a special type of tangents, the point of tangency being at infinity. In some special cases, however, the asymptotes are curvilinear so the comparison is not strictly true unless we can consider tangents as being curved also.

An envelope is defined(ii) as the curve or group of curves which are tangent to a family of curves. An envelope could never be an asymptote too but if it were possible for one member of the family of curves to have an asymptote this asymptote would closely approach the envelope at that point.

Evolutes and involutes are less closely related to asymptotes than any of the curves mentioned above.

(i) Edwards, Diff. Calculus, p.5; New Inter. Ency., vol. 19, p.21.
(ii) Granville, Diff. & Int. Calculus, p.205.

An evolute is the locus of the centers of curvature of a given curve and considering the evolute as the original curve the curve of which it is the evolute is called the involute.(i) Only in the case of a straight line would the evolute or involute pass to infinity and in that case there would be no asymptote.

IV. Kinds of Asymptotes.

There are three main kinds of asymptotes, two of which are closely related. The three kinds in order of their importance are:

A. Rectilinear. This includes all asymptotes which are straight lines and is the most common kind. Figures 3 to 7 are good examples of rectilinear asymptotes.

B. Curvilinear. "If there be two curves which continually approach each other so that for a common abscissa the limit of the difference of the ordinates is zero, or for a common ordinate the limit of the difference of the abscissa is zero, when that common abscissa or common ordinate is infinite, these curves are said to be asymptotic to each other".(ii) Any curve

(i) Edwards, Diff. Calculus, p.268; Granville, Diff. & Int. Calculus, p.182; New Inter. Ency., vol.7, p.321.

(ii) Edwards, Diff. Calculus, p.195; Davies & Peck, Dict. of Math., p.53; Frost, Curve Tracing, p.69.

therefore, of the general form

$$Y = AX^b + BX^{b-1} + \ldots + CX + D + EX^{-1} + \ldots$$

may have curvilinear asymptotes. For example, the curve

$$Y = AX^2 + BX + C + DX^{-1} + EX^{-2}$$

has the curvilinear asymptote

$$Y = AX^2 + BX + C$$

for the difference of their ordinates for any common abscissa is $DX^{-1} + EX^{-2}$, a quantity whose limit is zero when X is infinite.

C. Circular.(i) It sometimes happens that a spiral, drawn on polar coordinates, approaches a circle as the value of θ is increased indefinitely. That is, the value of r approaches some value, such as a, which is the radius of the circle. The circle is, therefore, the asymptote of the spiral and is called a circular asymptote. See figure 10.

V. The Curves Which Have Asymptotes.

It follows from the definitions of asymptotes that the following conditions are necessary and sufficient before a curve may have an asymptote. The two conditions are:(ii)

(i) Davies & Peck, Dict. of Math., p.54; Edwards, Diff. Calculus, p.205; Murray, Inf. Calculus, p.293.

(ii) Harding & Mullins, Ana. Geom., p.195; Murray, Inf. Calculus, p.286.

A. The curve must have an infinite branch.

B. The tangent to the curve at infinity, infinity is thought of as a region in this thesis, must pass within a finite distance of the origin (have a finite intercept on at least one of the coordinate axes).

The ways in which a curve may pass off to infinity are:(i)

a. X may be infinite, while Y is finite or zero.

b. Y may be infinite, while X is finite or zero.

Both X and Y may be infinite, dividing into three cases.

c. X and Y may be of the same order of magnitude, or X:Y finite.

d. X may be large compared with Y , or Y:X vanish when X and Y are increased indefinitely.

e. Y may be large compared with X, or X:Y vanish when X and Y are increased indefinitely.

Classes a and b include the cases in which the curve has asymptotes parallel to one or both of the coordinate axes.

Class c includes the curves which have rectilinear

(i) Frost, Curve Tracing, p.79.

asymptotes which are inclined at finite angles to the axes, and as special cases, parabolic asymptotes.

Classes d and e include curves which have curvilinear asymptotes.

If an equation in two variables be of odd degree it is proved in Theory of Equations that it must have at least one real root. Therefore any curve of odd degree, greater than the first degree, will have at least one real asymptote and will extend to infinity. No curve of odd degree can, therefore be closed nor can it have an even number of asymptotes.(i) Also a curve of even degree can not have an odd number of asymptotes.

Among the conics (curves which may be considered as plane sections of a cone) the hyperbola is the only one which has real asymptotes. The parabola is the only other conic with an infinite branch and the tangent to it at infinity does not pass within a finite distance of the origin.

VI. The Asymptotes of Hyperbolas.

The asymptotes of the hyperbola are the easiest to determine and are usually the only asymptotes discussed in elementary textbooks. They were the first to be determined by the Greeks and it was not till recent

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(i) Edwards, Diff. Calculus, p.184.

years that general methods for determining asymptotes were worked out.

Taking the standard form of the equation of the hyperbola

$$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1$$

and solving for Y we have

$$Y = \pm \frac{B}{A}\sqrt{X^2 - A^2}$$

Dividing both sides of the equation by X

$$\frac{Y}{X} = \pm \frac{B}{A}\sqrt{1 - \frac{A^2}{X^2}}$$

If X is now increased indefinitely the right hand member becomes $\pm \frac{B}{A}$. Hence as the distance from the origin increases the hyperbola approaches more and more closely, without ever reaching them, the lines

$$\frac{Y}{X} = \pm \frac{B}{A} \qquad \text{or} \qquad Y = \pm \frac{B}{A}X$$

These lines are, therefore, the asymptotes of the hyperbola.(i) This method is based on the third of the modern definitions given in section I.

The equations of the asymptotes of a hyperbola can also be found by considering the limiting form of the equation of the tangent as the point of tangency moves off to an infinite distance from the origin.

(i) Love, Ana. Geom., p.200; Woods & Bailey, A Course in Math., p.89.

The equation of the tangent of the hyperbola

$$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 1 \qquad (1)$$

at the point (X',Y') is

$$\frac{XX'}{A^2} - \frac{YY'}{B^2} = 1 \qquad (2)$$

Since the point (X',Y') is on the hyperbola we have

$$Y' = \pm \frac{B}{A}\sqrt{X'^2 - A^2} \qquad (3)$$

Substituting this value of Y' in (2) and dividing by X' we have

$$\frac{X}{A^2} \pm \frac{Y}{AB}\sqrt{1 - \frac{A^2}{X'^2}} = \frac{1}{X'} \qquad (4)$$

If the point of contact (X',Y') now moves off to an infinite distance from the origin so that X' becomes infinite, the limiting position of the line (4) is given by the equation

$$\frac{X}{A} \pm \frac{Y}{B} = 0 \qquad \text{or} \qquad Y = \pm \frac{B}{A}X$$

which is, therefore, the equation of the asymptotes.(i) This method is based on the second and sixth of the modern definitions given in section I.

There are many more methods(ii) for determining

(i) Candy, Pl. & Solid Ana. Geom., p.162-3.
(ii) Barnett, Pl. Ana. Geom., p.137; Harding & Mullins, Ana. Geom., p.195; Carmichael & Weaver, The Calculus, p.233; Siceloff-Wentworth-Smith, Ana. Geom., p.117.

the asymptotes of hyperbolas, some of which are more difficult than the two given above.

The equation of the asymptotes of the hyperbola may be written

$$\frac{X}{A} - \frac{Y}{B} = 0 \qquad \frac{X}{A} + \frac{Y}{B} = 0 \tag{5}$$

Multiplying these two equations together we obtain the degenerate equation(i)

$$\frac{X^2}{A^2} - \frac{Y^2}{B^2} = 0 \tag{6}$$

which represents the pair of asymptotes.(ii)

We immediately see that equation (6) differs from the equation of the hyperbola only in the constant term. Since we took the standard form of the equation of the hyperbola we may say that the equation of the asymptotes of a hyperbola differs from the equation of the hyperbola in the constant term only.

From the fact that the slopes of the asymptotes are $\pm \frac{B}{A}$ we derive the following result which gives a convenient method for drawing the asymptotes of any hyperbola whose axes are given.(iii)

The asymptotes of a hyperbola are the diagonal

(i) Siceloff-Wentworth-Smith, Ana. Geom., p.38
(ii) Candy, Pl. & Solid Ana. Geom., p.163; Love, Ana. Geom., p.201.
(iii) Love, Ana. Geom., p.121; Smith & Gale, New Ana. Geom., p.172; Siceloff-Wentworth-Smith, Ana. Geom., p.118.

lines of the rectangle whose center is the center of the curve and whose sides are parallel and equal to the axes of the curve. See figure 1.

If the axes of the hyperbola are equal (i. e. A = B) the equation of the asymptotes would be

$$Y = \pm X$$

and the asymptotes would be perpendicular to each other.(i) In this case the hyperbola is called an equilateral or rectangular hyperbola.

A special case of equilateral hyperbolas(ii) are those whose axes have been rotated so that their general equation is

$$XY = K$$

The coordinate axes are the asymptotes of this type of hyperbolas. This can be proved by solving the equation for X or Y and increasing the other variable indefinitely. In each case the hyperbola will approach an axis as a limit.

Example. Determine the asymptotes of the curve

$$X^2 - 4Y^2 = 16$$

Dividing the equation by 16 to put it in the standard form we have

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(i) Woods & Bailey, A Course in Math., p.90; Smith & Gale, New Ana. Geom., p.173.

(ii) Love, Ana. Geom., p.206.

$$\frac{X^2}{16} - \frac{Y^2}{4} = 1$$

The axes of the hyperbola are, therefore, equal to 4 and 2. Drawing the rectangle and then the asymptotes from these values we obtain the hyperbola and its asymptotes as shown in figure 2.

Substituting the values of A and B in the equation of the asymptotes we have

$$Y = \pm \frac{X}{2}$$

which is the equation of the asymptotes of figure 2.

We shall now consider methods for determining asymptotes of curves of any degree. The methods for obtaining the asymptotes of hyperbolas are applicable only when working with hyperbolas. The methods given below will also apply to hyperbolas.

VII. Method of Limiting Intercepts.(i)

The equation of the tangent to a curve at the point (X',Y') is

$$Y - Y' = \frac{dY'}{dX'}(X - X')$$

First placing Y =0 and solving for X, and

(i) Carmichael & Weaver, The Calculus, p.231; Townsend & Goodenough, Essentials of Calculus, p.145; Davies & Peck, Dict. Of Math., p.52; Murray, Inf. Calculus, p.291; Granville, Diff. & Int. Calculus, p.249.

then placing X = 0 and solving for Y, and denoting the intercepts by X_i and Y_i respectively, we obtain the equations

$$X_i = X' - Y'\frac{dX'}{dY'} \quad \text{and} \quad Y_i = Y' - X'\frac{dY'}{dX'}$$

Since the asymptote must pass within a finite distance of the origin one or both of these intercepts must have finite values as limits when the point of contact (X',Y') moves off to an infinite distance from the origin. If both limits are finite, that is, if

$$\text{limit}(X_i) = A \quad \text{and} \quad \text{limit}(Y_i) = B$$

then the equation of the asymptote is found by substituting the limiting values A and B in the equation

$$\frac{X}{A} + \frac{Y}{B} = 1$$

which is the intercept form of the equation of a straight line.

If only one of these limits exist but

$$\text{limit}\left(\frac{dY'}{dX'}\right) = M$$

then we have one intercept and the slope given so that the equation of the asymptote is

$$Y = MX + B \quad \text{or} \quad X = \frac{Y}{M} + A$$

In the case of curves of high degree this method is frequently too complicated to be very useful.

The method most commonly used is given in the next section.

VIII. The Substitution Method.(i)

Given the equation of the curve as

$$F(X,Y) = 0 \qquad (a)$$

where $F(X,Y)$ is a polynomial of the nth degree in X and Y.

Every tangent can be represented by the equation

$$Y = MX + B \qquad (b)$$

except those tangents which are parallel to the X axis. Substituting the value of Y given in equation (b) for Y in equation (a) we have

$$F(X, MX + B) = 0 \qquad (c)$$

Now the tangent is the limiting position of the secant as the two points of intersection of the secant with the curve approach coincidence. Hence for the tangent $Y = MX + B$ to be asymptotic to the curve $F(X,Y) = 0$ equation (c) must have two infinite roots, i.e., two points of intersection of the tangent with the curve must be at an infinite distance from

(i) Love, Ana. Geom., p.204; Carmichael & Weaver, The Calculus, p.233; Townsend & Goodenough, Essentials of Calculus, p.147; Cohen, Diff. & Int. Calculus, p.531; Candy, Pl. & Solid Ana. Geom., p.162; Murray, Inf. Calculus, p.290; Griffin, Math. Analysis, Higher Course, p.387; Granville, Diff. & Int. Calculus, p.252.

the origin. In Algebra it is proved that the condition for this is that the coefficients of X^n and X^{n-1} shall be zero. The proof for this is as follows:(i)

Let the given equation be

$$A_0X^n + A_1X^{n-1} + A_2X^{n-2} + \ldots + A_{n-1}X + A_n = 0 \qquad (1)$$

Substituting $\frac{1}{Z}$ for X in equation (1) gives

$$A_0(\tfrac{1}{Z})^n + A_1(\tfrac{1}{Z})^{n-1} + \ldots + A_{n-1}(\tfrac{1}{Z}) + A_n = 0 \qquad (2)$$

Multiplying equation (2) by Z^n gives

$$A_nZ^n + A_{n-1}Z^{n-1} + A_{n-2}Z^{n-2} + \ldots + A_1Z + A_0 = 0 \qquad (3)$$

If $A_n = 0$, one root of equation (1) is zero and hence the corresponding root of equation (3) is infinite (since $X = \frac{1}{Z}$).

Therefore, if the coefficient of the highest power of X in F(X) is equal to zero one root of F(X) is infinite. If the coefficients of the two highest powers of X are equal to zero then two roots of F(X) are infinite, and so on.

Returning to the general discussion; on equating these coefficients to zero we have two equations for determing the values of M and B. For every pair of values of M and B satisfying these equations we have, in the corresponding equation $Y = MX + B$, an asymptote to

(i) Murray, Inf. Calculus, p.286.

the curve $F(X,Y) = 0$.

To determine the asymptotes parallel to the Y-axis we substitute the equation $X = C$ in place of the equation $Y = MX + B$ and proceed in a similiar manner, equating the coefficients of Y^n and Y^{n-1} to zero.

It may happen that the term containing X^{n-1} is missing from equation (c), or the value of M obtained by placing the first coefficient equal to zero may cause the second coefficient to vanish. In this case the coefficients of X^n and X^{n-2} are equated to zero and we then have two equations for determining the values of M and B.

Example. Determine the asymptotes of the curve

$$X^3 + Y^3 - 3AXY = 0$$

Substituting $MX + B$ for Y we have

$$(1 + M^3)X^3 + (3M^2B - 3AM)X^2 + (3MB^2 - 3AB)X + B^3 = 0$$

Equating to zero the coefficients of X^3 and X^2 we have

$$1 + M^3 = 0 \qquad \text{or} \qquad M = -1$$

$$3M^2B - 3AM = 0$$

and substituting the value of M found above

$$3B + 3A = 0 \qquad \text{or} \qquad B = -A$$

Hence the asymptote is (figure 3)

$$Y = -X - A \qquad \text{or} \qquad Y + X + A = 0$$

If we substitute C for X we have

$$Y^3 + C^3 - 3ACY = 0$$

The coefficient of Y^3 can not be made zero and there is no term containing Y^2 so there is no asymptote parallel to the Y-axis.

Another substitution method(i) which is more difficult to prove but which is more easily applied in most cases is as follows:

Let the equation of any curve of the nth degree be arranged in homogeneous sets of terms and expressed as

$$X^n F_n\left(\frac{Y}{X}\right) + X^{n-1} F_{n-1}\left(\frac{Y}{X}\right) + X^{n-2} F_{n-2}\left(\frac{Y}{X}\right) + \dots = 0 \qquad \text{(d)}$$

To find where this curve is cut by any straight line whose equation is

$$Y = MX + B$$

substitute $M + \frac{B}{X}$ for $\frac{Y}{X}$ in equation (d) and the resulting equation is

$$X^n F_n\left(M + \frac{B}{X}\right) + X^{n-1} F_{n-1}\left(M + \frac{B}{X}\right) + X^{n-2} F_{n-2}\left(M + \frac{B}{X}\right) + \dots = 0 \qquad \text{(e)}$$

Applying Taylor's Theorem to expand each of these functional terms equation (e) becomes

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(i) Edwards, Diff. Calculus, p.182.

$$X^nF_n(M) + X^{n-1}\left|\begin{array}{l} BF'_n(M) \\ + F_{n-1}(M) \end{array}\right. + X^{n-2}\left|\begin{array}{l} \frac{B^2}{2}F''_n(M) \\ + BF'_{n-1}(M) \\ + F_{n-2}(M) \end{array}\right. + \ldots = 0 \qquad (f)$$

This is an equation of the nth degree proving that a straight line will, in general, intersect a curve of the nth degree in n points, real or imaginary.

The straight line $Y = MX + B$ is at our choice and therefore the two constants M and B may be chosen so as to satisfy any pair of consistent equations. If we choose M and B so that

$$F_n(M) = 0 \qquad (g)$$

and

$$BF'_n(M) + F_{n-1}(M) = 0 \qquad (h)$$

the two highest powers of X disappear from equation (f) and that equation will have two infinite roots.

If M_1, M_2, M_3, M_4, ..., M_n are the n values of M deduced from equation (g) the corresponding values of B will, in general, be given by the equations

$$B_1 = -\frac{F_{n-1}(M_1)}{F'_n(M_1)} \qquad (j)$$

$$B_2 = -\frac{F_{n-1}(M_1)}{F'_n(M_1)} \quad \text{and so on,}$$

and the n straight lines

$$Y = M_1X + B_1$$

$$Y = M_2X + B_2$$

$$\cdots$$

$$Y = M_nX + B_n$$

are the asymptotes of the curve.

Stated briefly this method, when applied to a problem, is: In the highest degree terms put $X = 1$ and $Y = M$ (this forms $F_n(M)$) and equate to zero. Hence find M. Form $F_{n-1}(M)$ in a similiar manner from the terms of the n-1 degree and differentiate $F_n(M)$. Then the several values of B are found by substituting the values of M in the equation

$$B = -\frac{F_{n-1}(M)}{F'_n(M)}$$

The corresponding values of M and B are then substituted in the equation

$$Y = MX + B$$

which gives the equations of the asymptotes.

Example. Determine the asymptotes of the curve

$$Y^3 - X^2Y + 2Y^2 + 4Y + X = 0$$

and the n straight lines

$$Y = M_1X + B_1$$

$$Y = M_2X + B_2$$

$$. \quad . \quad .$$

$$Y = M_nX + B_n$$

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Stated briefly this method, when applied to a problem, is: In the highest degree terms put $X = 1$ and $Y = M$ (this forms $F_n(M)$) and equate to zero. Hence find M. Form $F_{n-1}(M)$ in a similiar manner from the terms of the n-1 degree and differentiate $F_n(M)$. Then the several values of B are found by substituting the values of M in the equation

$$B = -\frac{F_{n-1}(M)}{F_n'(M)}$$

The corresponding values of M and B are then substituted in the equation

$$Y = MX + B$$

which gives the equations of the asymptotes.

Example. Determine the asymptotes of the curve

$$Y^3 - X^2Y + 2Y^2 + 4Y + X = 0$$

$$F_n(M) = M^3 - M = 0$$

Therefore $M = 0 \qquad M = \pm 1$

$$F_{n-1}(M) = 2M^2$$

$$F'_n(M) = 3M^2 - 1$$

$$B = -\frac{2M^2}{3M^2-1}$$

When $M = 0 \qquad B = 0$

$M = 1 \qquad B = -1$

$M = -1 \qquad B = -1$

Therefore the asymptotes are

$$Y = 0 \qquad Y = X - 1 \qquad Y + X + 1 = 0$$

IX. Number of Asymptotes to a Curve of the Nth Degree.(i)

Since the equation $F_n(M) = 0$ is generally of the nth degree in M, and the equation $BF'_n(M) + F_{n-1}(M) = 0$ is of the first degree in B, there are n values of M determined from the first equation and n corresponding values of B determined from the second equation. Hence it is possible for a curve of the nth degree to have n asymptotes. Some of these n asymptotes may be imaginary, i.e. not have a finite intercept on either of the coordinate axes, and in that case the curve would not have n real asymptotes.

(i) Edwards, Diff. Calculus, p.184.

If the term Y^n be missing from the equation of the curve, the term M^n will also be missing from the equation $F_n(M) = 0$ and there will be an apparent loss of degree in this equation. In this case, since the coefficient of M^n is zero, one root of the equation $F_n(M) = 0$ is infinite and therefore the corresponding asymptote is at right angles to the X-axis, i.e. parallel to the Y-axis. This leads to the special consideration of the asymptotes which are parallel to either of the coordinate axes.

X. Asymptotes Parallel to the Coordinate Axes.(i)

Let the equation of the curve be

$$A_0X^n + A_1X^{n-1}Y + A_2X^{n-2}Y^2 + \dots + A_{n-1}XY^{n-1} + A_nY^n$$
$$+ B_1X^{n-1} + B_2X^{n-2}Y + \dots + B_nY^{n-1} + C_2X^{n-2} + \dots = 0 \qquad (1)$$

If arranged in descending powers of X this is

$$A_0X^n + (A_1Y + B_1)X^{n-1} + (A_2Y^2 + B_2Y + C_2)X^{n-2} + \dots = 0 \qquad (2)$$

Now if A_0 vanish and Y be so chosen that

$$A_1Y + B_1 = 0$$

the coefficients of the two highest powers of X in equation (2) vanish and therefore two of the roots

(i) Edwards, Diff. Calculus, p.184; Townsend & Goodenough, Essentials of Calculus, p.148; Murray, Inf. Calculus, p.288; Griffin, Math. Analysis, p.387; Granville, Diff. & Int. Calculus, p.251.

are infinite. Hence the straight line $A_1Y + B_1 = 0$ is an asymptote of the curve.

If $A_0 = 0$, $A_1 = 0$, $B_1 = 0$, and if Y be so chosen that

$$A_2Y^2 + B_2Y + C_2 = 0$$

three roots of equation (2) are infinite and the lines represented by

$$A_2Y^2 + B_2Y + C_2 = 0$$

may be asymptotes to the curve, parallel to the X-axis.

Arranging equation (1) in descending powers of Y we have

$$A_nY^n + (A_{n-1}X + B_n)Y^{n-1} + \dots = 0$$

As above, if $A_n = 0$ then $A_{n-1}X + B_n = 0$ is an asymptote.

From the above results we deduce the following rule for finding the asymptotes parallel to the axes. Equate to zero the coefficients of the highest powers of X and Y.

Example. Determine the asymptotes of the curve

$$X^2Y^2 = C^2(X^2 + Y^2)$$

Rearranging terms

$$X^2Y^2 - C^2X^2 - C^2Y^2 = 0$$

Equating the coefficients of X^2 (the highest power of

X) to zero

$$Y^2 - C^2 = 0$$

The asymptotes parallel to the X-axis are, therefore,

$$Y = C \quad \text{and} \quad Y = -C$$

Equating the coefficients of Y^2 to zero

$$X^2 - C^2 = 0$$

and the asymptotes parallel to the Y-axis are

$$X = C \quad \text{and} \quad X = -C$$

The equation of the curve is of the fourth degree so we have determined all the possible asymptotes.

XI. Curves Whose Equation Is of the Form $Y = F(X)$.(i)

In some cases the equation of the curve may be of the general form $Y = F(X)$ and in this type there are special methods for determining the asymptotes which are simpler than the general methods given above.

If $X - C$ is a factor of $H(X)$ in

$$F(X) = \frac{G(X)}{H(X)} = \frac{A_0X^n + A_1X^{n-1} + \ldots + A_n}{B_0X^p + B_1X^{p-1} + \ldots + B_n} \qquad (1)$$

C being real, then the line $X - C = 0$ is an asymptote

(i) Cohen, Diff. & Int. Calculus, p.529; Fine, Calculus, p.52; Davies & Peck, Dict. of Math., p.53; Siceloff-Wentworth-Smith, Ana. Geom., p.345.

of the curve $Y = F(X)$, for, as X approaches C the value of Y approaches infinity.

If $n - p \gtreqless 1$ in equation (1) we can reduce $Y = F(X)$ to the form

$$Y = MX + B + K(X)$$

where M and B, one or both, may be zero, and K(X) denotes a proper fraction so that when X approaches infinity K(X) approaches zero. The curve will approach the line $Y = MX + B$ as the curve passes off to infinity and the line will, therefore, be an asymptote of the curve.

Curves whose equations are of the form $Y^2 = F(X)$ are solved in much the same manner as those of the form $Y = F(X)$.

If F(X) approaches infinity as X approaches some value, say B, then $X = B$ is an asymptote of the curve $Y^2 = F(X)$. Thus $X = 0$ is an asymptote of the curve $Y^2 = X + \frac{1}{X}$

The curve has oblique or horizontal asymptotes if F(X) can be reduced to the form $U^2 + V$ where U is real and of the first degree in X and $\frac{V}{U}$ approaches zero when X approaches infinity, the asymptotes being $Y = U$ and $Y = - U$.

For $\quad \pm Y = (U^2 + V)^{\frac{1}{2}} = |U| + \left[(U^2 \quad V)^{\frac{1}{2}} - |U|\right]$

$$= |U| + \frac{V}{(U^2 + V)^{\frac{1}{2}} + |U|}$$

and the fractional term approaches zero when $\frac{V}{U}$ approaches zero, therefore, when X approaches infinity.

This proof also shows that the equation

$$(Y - MX - C)^2 = U^2 + V \qquad \text{or} \qquad Y = MX + C \pm (U^2 + V)^{\frac{1}{2}}$$

has the asymptotes

$$Y = MX + C + U \qquad \text{and} \qquad Y = MX + C - U$$

Example. Determine the asymptotes of the curve

$$Y^2 = \frac{X^3 + X^2}{X - 1}$$

We see at once that $X - 1 = 0$ or $X = 1$ is an asymptote.

The equation may also be written

$$Y^2 = X^2 + 2X + 2 + \frac{2}{X - 1}$$

or $$Y^2 = (X + 1)^2 + \left(1 + \frac{2}{X - 1}\right)$$

Hence $\quad U = X + 1 \quad$ and $\quad V = 1 + \frac{2}{X - 1}$

$$\frac{V}{U} = \frac{1 + \frac{2}{X - 1}}{X + 1}$$

and this approaches zero as X approaches infinity so

the lines

$$Y = X + 1 \quad \text{and} \quad Y = -X - 1$$

are the asymptotes of the curve. See figure 4.

XII. Partial Fractions Method.(i)

The values of B given in equation (j), section VIII,

$$-\frac{F_{n-1}(M_1)}{F'_n(M_1)}, \quad -\frac{F_{n-1}(M_2)}{F'_n(M_2)}, \quad \text{etc.}$$

are exactly the constants required in putting

$$-\frac{F_{n-1}(T)}{F_n(T)}$$

into partial fractions.

For, suppose the single factor $(T - M_1)$ to occur in $F_n(T)$. Let

$$F_n(T) = (T - M_1)G(T)$$

Differentiating

$$F'_n(T) = G(T) + (T - M_1)G'(T)$$

and putting $T = M_1$

$$F'_n(T) = G(T)$$

But if $\frac{A}{T - M_1}$ be the partial fraction corresponding to the factor $(T - M_1)$

(i) Edwards, Diff. Calculus, p. 186.

$$A = -\frac{F_{n-1}(M_1)}{G(M_1)} = -\frac{F_{n-1}(M_1)}{F'_n(M_1)}$$

This gives an easy method of obtaining the asymptotes, for if

$$-\frac{F_{n-1}(T)}{F_n(T)} = \frac{B_1}{T - M_1} + \frac{B_2}{T - M_2} + \frac{B_3}{T - M_3} + \cdots$$

the asymptotes will be

$$Y = M_1X + B_1$$

$$Y = M_2X + B_2 \quad \text{, and so on.}$$

Example. Determine the asymptotes of the curve

$$(X^2 - Y^2)(X + 2Y) + 5(X^2 + Y^2) + X + Y = 0$$

$$\frac{F_{n-1}(T)}{F_n(T)} = \frac{5(T^2 + 1)}{(2T + 1)(T + 1)(T - 1)}$$

$$= \frac{-\frac{25}{3}}{2T + 1} + \frac{\frac{5}{3}}{T - 1} + \frac{5}{T + 1}$$

Hence the asymptotes are

$$2Y + X = -\frac{25}{3} \quad \text{or} \quad 6Y + 3X = -25$$

$$Y - X = \frac{5}{3} \quad \text{or} \quad 3Y - 3X = 5$$

$$Y + X = 5$$

XIII. Particular Cases of the General Theorem.[i]

We shall now return to a closer consideration of the equations

$$F_n(M) = 0 \qquad (g)$$

$$BF'_n(M) + F_{n-1}(M) = 0 \qquad (h)$$

of section VIII.

It is proved in Theory of Equations that if an equation such as $F_n(M) = 0$ has a pair of equal roots, say M_1, then $F'_n(M_1) = 0$.

There are three cases which we shall consider.

Case 1. Let the roots of $F_n(M) = 0$ be M_1, M_2, M_3, ..., M_n supposed all different so that $F'_n(M)$ does not vanish for any of these roots. Also suppose $F_n(M)$ and $F_{n-1}(M)$ to contain a common factor, say $M - M_1$, then $F_{n-1}(M_1) = 0$ and therefore $B_1 = 0$. Hence the corresponding asymptote is $Y = M_1X$ and it will pass through the origin.

Case 2. Next, suppose two of the roots of the equation $F_n(M) = 0$ are equal, say $M_1 = M_2$, then $F'_n(M_1) = 0$. In this case if $F_{n-1}(M)$ does not contain $M - M_1$ as one of its factors, the value of B determined

(i) Edwards, Diff. Calculus, p.186-7.

from equation (h) is infinite. The line $Y = M_1X + B_1$ does cut the curve in two points at infinity but it also makes an infinite intercept on the Y-axis and hence it lies wholly at an infinite distance from the origin. It does not, therefore, fulfill the requirements for an asymptote and will not be considered as such in this discussion.

Case 3. If $F_n(M) = 0$ has a pair of equal roots each equal to M_1, we have $F'_n(M) = 0$ and if M_1 is also a root of $F_{n-1}(M) = 0$, the value of B can not be determined from equation (h). We may, however, chose B so that the coefficient of X^{n-2} in equation (f) of section VIII vanishes, that is, so that

$$\frac{B^2}{2} F_n''(M) + BF'_{n-1}(M) + F_{n-2}(M) = 0$$

from which two values of B, real or imaginary, may be found. Let the roots of this equation be B_1 and B'_1. We thus obtain the equations of two parallel straight lines

$$Y = M_1X + B_1$$

$$Y = M_1X + B'_1$$

which each cut the curve in three points at infinity. In this case there is a double point on the curve at infinity. This double point is determined by a node

when $B_1 \neq B_1'$, a cusp when $B_1 = B_1'$

It is clear that in this case any straight line parallel to $Y = M_1X$ will cut the curve in two points at infinity. Of all this system of parallel straight lines the two whose equations we just found are the only ones which cut the curve in three points at infinity and therefore the name asymptote is confined to them. The one equation which includes both lines is obtained at once by substituting $Y - M_1X$ for B in the equation to obtain B and is

$$\frac{(Y - M_1X)^2}{2} F_n''(M_1) + (Y - M_1X)F_{n-1}'(M_1) + F_{n-2}(M_1) = 0$$

Example. Determine the asymptotes of the curve

$$X^3 + 2X^2Y + XY^2 - X^2 - XY + 2 = 0$$

Equating to zero the coefficients of Y^2 we obtain $X = 0$ the only asymptote parallel to either axis.

$$F_n(M) = 1 + 2M + M^2$$

$$M^2 + 2M + 1 = 0 \qquad \text{or} \qquad M = -1$$

$F_n(M)$ has a pair of equal roots so this problem comes under Case 3.

$$F_{n-1}(M) = -1 - M \qquad \text{and} \qquad F'(M) = 2 + 2M$$

Therefore

$$B = -\frac{-1 - M}{2 + 2M} = \frac{1}{2}$$

and the value of B_1 and B_2 can not be determined.

$$F_{n-2}(M) = 0 \qquad F''_n(M) = 2 \qquad F'_{n-1}(M) = -1$$

Substituting these values in

$$\frac{B^2}{2}F''_n(M) + BF'_{n-1}(M) + F_{n-2}(M) = 0$$

we have

$$\frac{B^2}{2}(2) + B(-1) + 0 = 0$$

$$B^2 - B = 0$$

$$B = 0$$

$$B = 1$$

Hence the asymptotes are

$$Y = -X + 0 \qquad \text{or} \qquad Y + X = 0$$

$$Y = -X + 1 \qquad\qquad Y + X = 1$$

$$X = 0$$

the first two of which are parallel.

XIV. Obtaining the Asymptotes by Determining the Form of the Equation at Infinity.(i)

P_r and F_r will be used to denote rational algebraic expressions which contain terms of the rth degree and lower, but contain no terms of higher degree than r.

Suppose the equation of a curve of the nth

(i) Edwards, Diff. Calculus, p. 183.

degree to be of the form

$$(AX + BY + C)P_{n-1} + F_{n-1} = 0 \qquad (1)$$

Then any straight line parallel to $AX + BY = 0$ cuts the curve in one point at infinity. To find the particular member of this family of parallel straight lines which cuts the curve in a second point at infinity we shall consider the ultimate linear form to which the curve approximates as we travel to infinity in the above direction, thus obtaining the ultimate direction of the curve and forming the equation of the tangent at infinity. To do this we make X and Y of the curve become large in the ratio given by $X:Y = -B:A$, that is in a certain direction whose slope is $-\frac{A}{B}$, and we obtain the equation

$$AX + BY + C + \lim_{Y = -\frac{A}{B}X = \infty} \frac{F_{n-1}}{P_{n-1}} = 0 \qquad (2)$$

If this limit be finite we have arrived at the equation of a straight line which, at infinity, represents the limiting form of the curve and which satisfies the definition of an asymptote.

To obtain the value of the limit it is advantageous to put $X = -\frac{B}{T}$ and $Y = \frac{A}{T}$ and after simplification make $T = 0$.

Example. Determine the asymptotes of the curve

$$(X + Y)(X^4 + Y^4) = A(X^4 + A^4)$$

$$X + Y = \lim_{X = -Y = \infty} \frac{A(X^4 + A^4)}{(X^4 + Y^4)}$$

Substituting $X = -\frac{1}{T}$ and $Y = \frac{1}{T}$

$$X + Y = \lim_{T = 0} \frac{A(\frac{1}{T^4} + A^4)}{(\frac{1}{T^4} + \frac{1}{T^4})}$$

$$X + Y = \lim_{T = 0} \frac{A + A^4T^4}{2} = \frac{A}{2}$$

Therefore $X + Y = \frac{A}{2}$ is an asymptote.

Next, suppose the equation of the curve to be of the form

$$(AX + BY + C)F_{n-1} + F_{n-2} = 0 \tag{3}$$

Then the line $AX + BY + C = 0$ cuts the curve in two points at infinity for no terms of the n or n-1 degree remain in the equation determining the points of intersection. Hence, in general, the line

$$AX + BY + C = 0$$

is an asymptote. We say in general for if F_{n-1} be of the form $(AX + BY + C)P_{n-2}$ itself containing a factor $(AX + BY + C)$ there will be, as in Case 3, section XIII, a pair of asymptotes parallel to AX BY C = 0, each cutting the curve in three points at infinity. The equation of the curve then becomes.

$$(AX + BY + C)^2P_{n-2} + F_{n-2} = 0 \tag{4}$$

and the equations of the parallel asymptotes are

$$AX + BY + C = \pm\sqrt{\operatorname{limit} - \frac{F_{n-2}}{P_{n-2}}}$$

where X and Y in the limit on the right hand side become infinite in the ratio X:Y = - B:A.

Or, if the equation of the curve be written in the form

$$(AX + BY)^2 P_{n-2} + (AX + BY) F_{n-2} + f_{n-2} = 0 \qquad (5)$$

in proceeding to infinity in the direction $AX + BY = 0$ we have

$$(AX + BY)^2 + (AX + BY) \operatorname{limit} \frac{F_{n-2}}{P_{n-2}} + \operatorname{limit} \frac{f_{n-2}}{P_{n-2}} = 0$$

and the limits are to be obtained by putting $X = -\frac{B}{T}$ and $Y = \frac{A}{T}$ and then diminishing T indefinitely. We thus obtain a pair of parallel asymptotes

$$AX + BY = D$$

$$AX + BY = E$$

where D and E are the roots of the equation

$$R^2 + R \operatorname{limit} \frac{F_{n-2}}{P_{n-2}} + \operatorname{limit} \frac{f_{n-2}}{P_{n-2}} = 0$$

Other particular forms which the equations of curves may assume can be treated similiarly.

Example. Determine the asymptotes of the curve

$$(2X - 3Y + 1)^2(X + Y) - 8X + 2Y - 9 = 0$$

$$2X - 3Y + 1 = \pm\sqrt{\text{limit}\,\frac{8X - 2Y + 9}{X + Y}}$$

and putting in $X = \frac{3}{T}$ and $Y = \frac{2}{T}$

$$2X - 3Y + 1 = \pm\sqrt{\lim_{T=0} \frac{\frac{24}{T} - \frac{4}{T} + 9}{\frac{3}{T} + \frac{2}{T}}}$$

$$2X - 3Y + 1 = \pm\sqrt{\lim_{T=0} \frac{20 + 9T}{5}} = \pm 2$$

Therefore the asymptotes, which are parallel lines, are

$$2X - 3Y = 1$$

$$2X - 3Y = -3$$

XV. When the Equation of the Curve Has Linear Factors.(i)

If in the equations of the general form

$$F_n + F_{n-2} = 0 \tag{1}$$

F_n breaks up into n linear factors so as to represent n straight lines, no two of which are parallel, these straight lines, whose equation will be

$$F_n = 0 \tag{2}$$

(i) Edwards, Diff. Calculus, p.190; Frost, Curve Tracing, p.100.

will be the asymptotes of the curve whose equation is of the form given in equation (1). For, as the curve approaches any of the n straight lines given in equation (2) the terms containing the two highest powers will vanish and there will be two intersections of the line with the curve at infinity.

Example. Determine the asymptotes of the curve

$$Y(Y + 2X)(Y - X)^2 = 3C^2X^2$$

We see at once that the lines

$$Y = 0$$

$$Y + 2X = 0$$

are asymptotes. The other asymptotes are parallel to $Y - X = 0$ and by section XIV they are

$$Y - X = \pm\sqrt{\text{limit}\ \frac{3C^2X^2}{Y^2 + 2XY}}$$

Substituting $X = -\frac{1}{T}$ and $Y = -\frac{1}{T}$

$$Y - X = \pm\sqrt{\lim_{T=0} \frac{\frac{3C^2}{T^2}}{\frac{1}{T^2} + \frac{2}{T^2}}} = \pm C$$

The parallel asymptotes are

$$Y - X = C$$

$$Y - X = -C$$ See figure 5.

If the equation of the curve is arranged in homogeneous sets of terms such as(i)

(i) Edwards, Diff. Calculus, p.191; Frost Curve Tracing, p.71.

$$U_n + U_{n-2} + U_{n-3} + \dots = 0 \qquad (3)$$

and it be found that there are no terms of the n-1 degree and that U_n contains no repeated factors, then the n straight lines given by equation

$$U_n = 0 \qquad (4)$$

are the asymptotes of the curve whose equation is of the general form given in equation (3). These lines will all pass through the origin for there will be no constant term in equation (4).

Example. Determine the asymptotes of the curve

$$(Y - X)^2(Y + X)(Y + 2X) = 16A^4$$

This equation is of the general form given in equation (3) so the asymptotes are given by the equation

$$(Y - X)^2(Y + X)(Y + 2X) = 0$$

and they are

$$Y = X$$

$$Y = -X$$

$$Y = -2X \qquad \text{See figure 6.}$$

While it is true that in this problem U_n contains a repeated factor $(Y - X)$ it does not lead to parallel asymptotes as would be expected. This is due to the fact that there are no terms containing the variables of any power lower than the fourth which is also the highest power. The

curve approaches this asymptote ($Y = X$) from both sides at each end so it really takes the place of two asymptotes.

XVI. Intersections of a Curve With Its Asymptotes.[i]

If a curve of the nth degree has n asymptotes, no two of which are parallel, we have seen (in section XV) that the equations of the asymptotes and of the curve may be written

$$F_n = 0 \qquad (1)$$

$$F_n + F_{n-2} = 0$$

The n asymptotes therefore intersect the curve again at points lying on the curve

$$F_{n-2} = 0 \qquad (2)$$

Each asymptote cuts its curve in two points at infinity and also in n - 2 other points given by equation (2). These n(n - 2) points of intersection will always lie on a certain curve of the n - 2 degree. For example:

The asymptotes of a cubic will cut the curve again in three points lying in a straight line.

The asymptotes of a quartic will cut the curve again in eight points lying on a conic. (Important facts in the theory of cubics and quartics)

(i) Edwards, Diff. Calculus, p.191; Frost, Curve Tracing, p.71.

And so on for curves of higher degree.

XVII. Position of the Asymptote With Respect to the Curve.

Suppose the straight line $AX + BY + C = 0$ to be an asymptote of a curve and that there is no other asymptote of the curve parallel to this. The equation of the curve will be of the form

$$(AX + BY + C)F_{n-1} + F_{n-2} = 0$$

and the perpendicular from any point (X,Y) of the curve on this asymptote is

$$P = \frac{1}{\sqrt{A^2 + B^2}} \frac{F_{n-2}}{F_{n-1}}$$

When X and Y become very large in the ratio $X:Y = -B:A$ this may be written

$$P = \frac{k}{X}f(\frac{Y}{X})$$

where k is a constant. P therefore changes signs the same as X and in general the curve at opposite extremities of the asymptote lies on opposite sides of it.

If $AX + BY$ is a factor of the terms of the highest degree in F_{n-2} the equation of the curve may be written

$$(AX + BY + C)F_{n-1} + F_{n-3} = 0$$

and the perpendicular on the asymptote is

(i) Edwards, Diff. Calculus, p.194; Frost, Curve Tracing, p.80.

$$P = \frac{AX + BY + C}{\sqrt{A^2 + B^2}} = -\frac{1}{\sqrt{A^2 + B^2}} \frac{F_{n-3}}{F_{n-1}}$$

and when X and Y are increased indefinitely in the ratio $X:Y = -B:A$ this may be written

$$P = \frac{k}{X^2} f\left(\frac{Y}{X}\right)$$

and this does not change signs with X so in this case the curve lies on the same side of the asymptote at opposite extremities of it.

If the equation of the curve is of the form

$$(AX + BY + C)^2 F_{n-2} + F_{n-2} = 0$$

(the same as equation (4), section XIV) the expression for the length of the perpendicular is in the limit of the form $f(\frac{Y}{X})$. Generally this will not vanish as the curve has parallel asymptotes and will as a rule lie between them.

When the equation of the curve is of the form

$$(AX + BY + C)^2 F_{n-2} + F_{n-3} = 0$$

the length of the perpendicular is

$$P^2 = -\frac{1}{\sqrt{A^2 + B^2}} \frac{F_{n-3}}{F_{n-2}}$$

and when X and Y become very large in the ratio $X:Y = -B:A$ this may be written

$$P = \pm\sqrt{\frac{k}{X} f\left(\frac{Y}{X}\right)}$$

X can not change signs or the perpendicular will become imaginary at one extremity of the asymptote. The line is, therefore, asymptotic at only one extremity and the curve approaches it from both sides there.

When the equation of the curve is of the same general form as those discussed in section XI the position of the curve with respect to the asymptote is readily found. It is sometimes easier to determine the position of the curve with respect to the asymptote if the equation of the curve is changed to this form.

To change the general equation[i]

$$X^nF_n\left(\frac{Y}{X}\right) + X^{n-1}F_{n-1}\left(\frac{Y}{X}\right) + X^{n-2}F_{n-2}\left(\frac{Y}{X}\right) + \ldots = 0 \qquad (1)$$

to the form

$$Y = AX + B + \frac{C}{X} + \frac{D}{X^2} + \ldots \qquad (2)$$

substitute the value of Y given in equation (2) in equation (1). Since the result must be an identity the coefficients of each power of X will be equal to zero. This will give enough equations to determine A, B, C, D, etc.

The result of the above substitution (after expansion) is

(i) Edwards, Diff. Calculus, p. 198.

$$X^nF_n(A) + X^{n-1}\left|\begin{array}{l} BF'_n(A) \\ \\ +F_{n-1}(A) \\ \\ \\ \\ \end{array}\right. + X^{n-2}\left|\begin{array}{l} CF'_n(A) \\ \\ +\frac{B^2}{2}F''_n(A) \\ \\ +BF'_{n-1}(A) \\ \\ +F_{n-2}(A) \end{array}\right. + \dots = 0$$

which gives the equations

$$F_n(A) = 0 \qquad\qquad BF'_n(A) \quad F_{n-1}(A) = 0$$

$$CF'_n(A) + \frac{B^2}{2}F''_n(A) + BF'_{n-1}(A) + F_{n-2}(A) = 0$$

from which the constants A, B, C, etc. can be determined.

As in section XI, if the equation of the curve is

$$Y = AX + B + \frac{C}{X} + \frac{D}{X^2} + \cdots$$

the equation of the asymptote will be

$$Y = AX + B$$

Let Y be the ordinate of the curve and Y' be the ordinate of the asymptote; then

$$Y - Y' = \frac{C}{X} + \frac{D}{X^2} + \cdots$$

and as X becomes very large the sign of C governs the sign of the right side.

If X and Y are positive, i.e. in the first quadrant, Y - Y' will have, in the limit, the same

sign as C. When C is positive Y - Y' will be positive and the curve will approach the asymptote from above. Similiarly, if C is negative Y - Y' will be negative and the curve will approach the asymptote from below. In the same manner the position of the curve with respect to the asymptotes can be determined in the other quadrants.

Example. Determine the asymptotes and the position of the curve with respect to them of the curve

$$(Y - X)^2X - 3Y(Y - X) + 2X = 0$$

The coefficients of Y^2 is $X - 3$, therefore $X = 3$ is an asymptote

The equation of the curve may also be written as

$$(Y - X)^2 - 3(Y - X)\frac{Y}{X} + 2 = 0$$

and in the direction $Y = X$ at infinity this ultimately becomes (equation 5, section XIV)

$$(Y - X)^2 - 3(Y - X) + 2 = 0$$

and therefore

$$Y - X = 1 \qquad \text{and} \qquad Y - X = 2$$

are asymptotes of the curve.

Substituting $A + \frac{B}{X} + \ldots$ for $Y - X$ in the equation of the curve we have

$$(A + \frac{B}{X} + \ldots)^2 X - 3(X + A + \frac{B}{X} + \ldots)(A + \frac{B}{X} + \ldots) + 2X = 0$$

$$X(A^2 - 3A + 2) + 2AB - 3(A^2 + B) = 0$$

Equating coefficients to zero

$$A^2 - 3A + 2 = 0 \quad \text{or} \quad A = 1 \text{ or } 2$$

$$2AB - 3A^2 - 3B = 0 \quad \text{or} \quad B = -3 \text{ or } 12$$

Therefore the equation of the curve may be written in either of the forms

$$Y = X + 1 - \frac{3}{X}$$

$$Y = X + 2 + \frac{12}{X}$$

Hence to the right of the Y-axis the curve lies below the asymptote $Y = X + 1$ and above the asymptote $Y = X + 2$. On the left of the Y-axis the curve is above the asymptote $Y = X + 1$ and below $Y = X + 2$. See figure 7.

XVIII. Curvilinear Asymptotes.(i)

As given in section IV, any curve of the general form

$$Y = AX^b + BX^{b-1} + \ldots + CX + D + EX^{-1} + \ldots$$

may have curvilinear asymptotes. An example of this is the curve

$$Y = AX^2 + BX + C + DX^{-1} + EX^{-2}$$

(i) Edwards, Diff. Calculus, p.195; 199-202; Frost, Curve Tracing, p.100; Davies & Peck, Dict. of Math., p.54.

which has the curvilinear asymptote

$$Y = AX^2 + BX + C$$

Returning to the general form of the equation as given in equation (d), section VIII, If $F_n(M)$ has equal roots, say M_1, then $F'_n(M_1) = 0$ (section XIII). If $F_{n-1}(M_1)$ does not also vanish the value of B can not be determined. The equation of the curve would be of the form

$$(Y - M_1X)^2V_{n-2} + U_{n-1} + U_{n-2} + \dots + U_0 = 0$$

where U_{n-1} does not contain the factor $Y - M_1X$. We can write this as

$$(Y - M_1X) + \frac{U_{n-1}}{V_{n-2}} + \frac{U_{n-2}}{V_{n-2}} + \dots = 0$$

and if we put A for limit $\frac{U_{n-1}}{XV_{n-2}}$ and B for limit $\frac{U_{n-2}}{V_{n-2}}$ when X and Y become very large in the ratio $1:M_1$ the curve approximates to the parabolic form

$$(Y - M_1X)^2 + AX + B = 0$$

This parabola, although an approximation to the shape of the curve, is not generally asymptotic to it. It suggests that in closely examining the parabolic branches we should try to expand Y in the form

$$\frac{Y}{X} = M + \frac{A}{X^{\frac{1}{2}}} + \frac{B}{X} + \frac{C}{X^{3/2}} + \frac{D}{X^2} + \dots$$

If we substitute this in the equation

$$X^nF_n(\tfrac{Y}{X}) + X^{n-1}F_{n-1}(\tfrac{Y}{X}) + X^{n-2}F_{n-2}(\tfrac{Y}{X}) + \dots = 0$$

and expand as before the result (after collecting the coefficients of like powers of X) is

$$X^nF_n(M)$$

$$+ X^{n-\frac{1}{2}}\left(AF_n'(M)\right)$$

$$+X^{n-1}\left(BF_n'(M) + \tfrac{A^2}{2}F_n''(M) + F_{n-1}(M)\right)$$

$$+X^{n-\frac{3}{2}}\left(CF_n'(M) + ABF_n''(M) + \tfrac{A^3}{6}F_n'''(M) + AF_{n-1}'(M)\right)$$

$$+X^{n-2}\left(DF_n'(M) + \tfrac{B^2}{2}F_n''(M) + ACF_n''(M) + \tfrac{A^2B}{2}F_n'''(M) + \tfrac{A^4}{24}F_n''''(M) + BF_{n-1}'(M) + \tfrac{A^2}{2}F_{n-1}''(M) + F_{n-2}(M)\right)$$

$$+\dots = 0$$

and equating to zero the coefficients we have the equations

$F_n(M) = 0$ (by supposition $F_n'(M) = 0$)

$$\tfrac{A^2}{2}F_n''(M) + F_{n-1}(M) = 0$$

$$BF_n''(M) + \tfrac{A^2}{6}F_n'''(M) + F_{n-1}'(M) = 0$$

$$ACF_n''(M) + \tfrac{B^2}{2}F_n''(M) + \tfrac{A^2B}{2}F_n'''(M) + \tfrac{A^4}{24}F_n''''(M) + BF_{n-1}'(M) + \tfrac{A^2}{2}F_{n-1}''(M) + F_{n-2}(M) = 0$$

etc.

which will determine the constants A, B, C, etc.

The parabola $(Y - MX - B)^2 = A^2X + 2AC$

is then asymptotic to the curve.

In practice it is found more convenient to use a method of successive approximation to obtain the ultimate form of the parabolic branch at infinity. This method is easier to explain by using an example.

Example. Determine the asymptotes of the curve

$$(Y - X)^2(Y + X) = 2AX^2$$

The rectilinear asymptote is parallel to $Y + X = 0$ (section XIV) and rewriting the equation in the form

$$Y + X = \frac{2AX^2}{(Y - X)^2}$$

Putting $X = -\frac{1}{T}$ and $Y = \frac{1}{T}$

$$Y + X = \lim_{T=0} \frac{\frac{2A}{T^2}}{\frac{4}{T^2}} = \frac{A}{2}$$

Therefore $2X + 2Y = A$ is an asymptote.

To obtain a closer approximation solve the equation for Y and substitute the value found in the right hand member of the equation.

$$Y + X = \frac{2AX^2}{(\frac{A}{2} - 2X)}$$

$$= \frac{A}{2}(1 - \frac{A}{4X})^{-2}$$

$$= \frac{A}{2} + \frac{A^2}{4X}$$

Hence to the right of the Y-axis the curve lies above the asymptote and to the left of the Y-axis it lies below the asymptote.

The axis of the parabolic asymptote is in the direction Y = X. For the first approximation to the shape at infinity we have

$$Y - X = \sqrt{\frac{2AX^2}{Y + X}} = \sqrt{\frac{2AX^2}{X + X}} = \sqrt{AX} \qquad (1)$$

For the second approximation substitute this value of Y and we obtain

$$Y - X = \sqrt{\frac{2AX^2}{X + X + \sqrt{AX}}} = \sqrt{\frac{2AX}{2 + \sqrt{\frac{A}{X}}}}$$

$$= \sqrt{AX}\left(1 + \frac{1}{2}\sqrt{\frac{A}{X}}\right)^{\frac{1}{2}} = \sqrt{AX}\left(1 - \frac{1}{4}\sqrt{\frac{A}{X}} + \ldots\right)$$

$$Y - X = \sqrt{AX} - \frac{A}{4} \qquad (2)$$

For a third approximation use the value of Y given by (2) and substitute it in (1).

$$Y - X = \sqrt{\frac{2AX^2}{X + X + \sqrt{AX} - \frac{A}{4}}}$$

$$= \sqrt{AX}\left(1 + \frac{1}{2}\sqrt{\frac{A}{X}} - \frac{A}{8X}\right)^{\frac{1}{2}}$$

$$Y - X = \sqrt{AX} - \frac{A}{4} + \frac{5A}{32X^{\frac{1}{2}}} \qquad (3)$$

Although the first approximation, equation (1), indicates the shape of the curve at infinity it is not asymptotic to the curve.

The second approximation, equation (2), gives a parabola which, as is seen from equation (3), is such that the limit of the difference of its ordinate from that of the curve is zero, and though not itself being that parabola which most closely approximates the shape of the curve at infinity, is nevertheless useful in tracing the curve. This is the parabola given in figure 8, the figure for this example.

The third approximation, equation (3), shows that the ordinate of the upper branch of the parabola is less than that of the curve and that the ordinate of the lower branch of this parabola is greater than that of the curve. Both branches of the curve, therefore, approach the parabolic asymptote from the outside.

Equation (3) also shows that the true asymptotic parabola is

$$(Y - X + \frac{A}{4})^2 = AX + \frac{5A^2}{16}$$

which is coaxial with the parabola, equation (2), which is shown in figure 8.

XIX. Polar Asymptotes.[i]

Let the equation of the curve be

$$R^nF_n(\theta) + R^{n-1}F_{n-1}(\theta) + \dots + F_0(\theta) = 0 \qquad (1)$$

or

$$U^nF_0(\theta) + U^{n-1}F_1(\theta) + \dots + F_n(\theta) = 0 \qquad (2)$$

To find the directions in which $R = \infty$ or $U = 0$ we have

$$F_n(\theta) = 0 \qquad (3)$$

and let the roots of this equation be $\theta = a, b, c$, etc.

Let angle XOP $= a$, (see figure 9). Then the radius OP, the curve, and the asymptote meet at infinity towards P. Let OY ($= p$) be the perpendicular upon the asymptote. Since OY is at right angles to OP it is the polar subtangent, and $p = -\frac{d\theta}{dU}$. Let angle XOY $= a'$ and let Q be any point on the asymptote whose coordinates are R, θ. Then the equation of the asymptote is

$$p = R\cos(\theta - a') \qquad (4)$$

and it can readily be seen from the figure that $a' = a - \frac{\pi}{2}$.

To find the value of $-\frac{d\theta}{dU}$ when $U = 0$ differ-

(i) Edwards, Diff. Calculus, p.203; Granville, Diff. & Int. Calculus, p.254; Woods & Bailey, A Course in Math., p.332; Davies & Peck, Dict. of Math., p.53; Carmichael & Weaver, The Calculus, p.235; Murray, Inf. Calculus, p.292.

differentiate equation (2), put $U = 0$ and $\theta = a$ and we have

$$\left(\frac{dU}{d\theta}\right)_{U=0} F_{n-1}(a) + F'_n(a) = 0 \qquad (5)$$

Substituting the value of $\left(-\frac{dU}{d\theta}\right)_{U=0}$ found in equation (5) in equation (4) we have

$$\frac{F_{n-1}(a)}{F'_n(a)} = R\cos(\theta - a + \frac{\pi}{2}) = R\sin(a - \theta)$$

Therefore the equations of the asymptotes are

$$R\sin(a - \theta) = \frac{F_{n-1}(a)}{F'_n(a)}$$

$$R\sin(b - \theta) = \frac{F_{n-1}(b)}{F'_n(b)} \qquad , \text{ etc.}$$

The case most often met with is that in which $n = 1$ and the equation of the curve is of the form

$$RF_1(\theta) + F_0(\theta) = 0$$

Then $F_1(\theta) = 0$ gives a, b, c, etc. and the asymptotes are

$$R\sin(a - \theta) = \frac{F_0(a)}{F'_1(a)} \qquad , \text{ etc.}$$

The equivalent Cartesian form is

$$Y = X\tan a + \sec a\left(\frac{d\theta}{dU}\right)_{\theta=a}$$

or

$$Y\cos a = X\sin a + \left(\frac{d\theta}{dU}\right)_{U=0}$$

and will be found more convenient in some cases.

The following rule for drawing the asymptotes will be found very useful. After having found the values of $\left(\frac{d\theta}{dU}\right)_{U=0}$ suppose we stand at the origin and look in the direction of that value of θ which makes $U = 0$. Draw a line at right angles to that direction through the origin and of length equal to $\left(-\frac{d\theta}{dU}\right)_{U=0}$ and to the right or left according as the value is positive or negative. Through the end of this line draw a perpendicular to it of indefinite length. This straight line will be the asymptote.

Example. Determine the asymptotes of the curve

$$R = A \tan \theta$$

or

$$R \cos \theta - A \sin \theta = 0$$

$$F_1(\theta) = \cos \theta \qquad \text{and} \qquad F_0(\theta) = - A \sin \theta$$

$$\cos \theta = 0 \quad \text{gives } a = \frac{\pi}{2} \qquad b = \frac{3\pi}{2}$$

$$\text{and} \quad \frac{F_0(a)}{F_1'(a)} = \frac{- A \sin a}{- \sin a} = A$$

Hence the asymptotes are

$$R \sin\left(\frac{\pi}{2} - \theta\right) = A \qquad \text{or} \qquad R \cos \theta = A$$

$$R \sin\left(\frac{3\pi}{2} - \theta\right) = A \qquad \text{or} \qquad R \cos \theta = - A$$

Using the Cartesian formula

$$U = \frac{1}{A} \cot \theta$$

$$U = 0 \qquad \text{when} \qquad \theta = n\pi + \frac{\pi}{2}$$

$$\text{and} \quad -\frac{d\theta}{dU} = A \sin^2 \theta = A$$

Therefore the equation $Y \cos a = X \sin a + (\frac{d\theta}{dU})U{=}0$ becomes

$$X = \pm A$$

XX. Circular Asymptotes.[i]

In some polar equations when θ is increased indefinitely the equation takes the form of an equation in R which represents one or more concentric circles.

For example, in the curve

$$R = A\frac{\theta}{\theta - 1} \qquad \text{or} \qquad R = A\frac{1}{1 - \frac{1}{\theta}}$$

it is clear that if θ becomes very large the curve approaches indefinitely near the limiting circle

$$R = A \qquad \text{(See figure 10)}$$

Such a circle is called an asymptotic circle or circular asymptote of the curve.

(i) Murray, Inf. Calculus, p.292; Davies & Peck, Dict. of Math., p.54; Edwards, Diff. Calculus, p.205.

XXI. Asymptotes of Three Dimensional Figures.(i)

Asymptotes of three dimensional figures are, as a rule, very hard to determine and there has been very little work done with them except in the case of hyperboloids.

Let the equation of the hyperboloid be

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} - \frac{Z^2}{C^2} = 1 \qquad (1)$$

and let

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} - \frac{Z^2}{C^2} = 0 \qquad (2)$$

be the equation of a cone along the Z-axis.

The equations of the contours of these surfaces made by the plane Z = K are respectively

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = 1 + \frac{K^2}{C^2} \qquad (3)$$

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = \frac{K^2}{C^2} \qquad (4)$$

A comparison of equations (3) and (4) shows that for the same finite value of K the section of the cone is smaller than the corresponding section of the hyperboloid. Hence the cone may be said to lie inside the hyperboloid.

(i) Candy, Pl. & Solid Ana. Geom., p.231-2; Siceloff- Wentworth-Smith, Ana. Geom., p.173.

Equation (3) may be written

$$\frac{X^2}{A^2} + \frac{Y^2}{B^2} = \frac{K^2}{C^2}(1 + \frac{C^2}{K^2})$$

which shows that the sections of the two surfaces become equal, i.e., they approach the same limit, when the cutting plane recedes in either direction to an infinite distance from the origin. That is, the cone is tangent to the hyperboloid at infinity and is, therefore, called an asymptotic cone of the hyperboloid of one sheet.

The asymptotic cone of the hyperboloid of two sheets can be found in much the same manner and the equation of the cone is

$$\frac{X^2}{A^2} - \frac{Y^2}{B^2} - \frac{Z^2}{C^2} = 0$$

XXII. Summary.

Asymptotes are very useful in drawing the graphs of plane curves, also known as curve tracing. One of the most important rules for curve tracing is: "Examine the curve for infinite branches and the asymptotes to them (if such exist)"(i) By first determining the asymptotes the time required for drawing the curve is materially lessened. Especially is this true if

(i) Carmichael & Weaver, The Calculus, p.236.

the equation of the curve is of a high degree when, without using asymptotes, the curve would be very hard to graph.

The form of the curve at infinity can also be studied by using asymptotes and double points at infinity determined. Any small branch of the curve, such as the one in figure 7, is not so apt to be missed if the asymptotes of the curve are drawn.

A comparison of euclidean and non-euclidean geometry shows some interesting facts concerning asymptotes and asymptotic lines.

"In euclidean geometry the asymptotes are tangents to the conic at the points where it cuts the absolute; but in non-euclidean geometry the lines which most closely resemble the euclidean asymptotes are the tangents to the conic from the center, and are, therefore, six in number"(i)

"In non-euclidean geometry equidistant straight lines can not exist. Intersectors are convergent or divergent in the same sense as in euclidean geometry; parallel are convergent and asymptotic in one direction and divergent in the other; non-intersectors are ultimately divergent in both directions"(ii)

(i) Sommerville, Non-Euclidean Geom., p.259-60.
(ii) ibid., p.42.

These quotations show that there is no relation between the two geometries in the matter of asymptotes and as this thesis was written from the euclidean standpoint any work based on non-euclidean geometry was omitted.

Various problems often occur in Physics and Engineering which are made clearer by graphing and drawing the asymptotes. Probably the most common is the equilateral hyperbola (section V) in which the coordinate axes are the asymptotes of the curve.

The word asymptote or asymptotic is often used to designate two things, other than lines, which continually approach each other or are very nearly alike. Asymptotic series in infinite series are an example of this.

Imaginary asymptotes were not discussed for they are of no help in drawing a curve and they do not have a finite intercept on either of the coordinate axes. Hence they are not true asymptotes as asymptotes have been defined in this discussion (section V).

Figures

(Asymptotes are drawn in red)

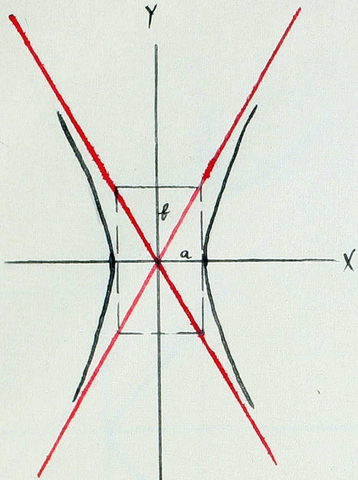

Figure 1.

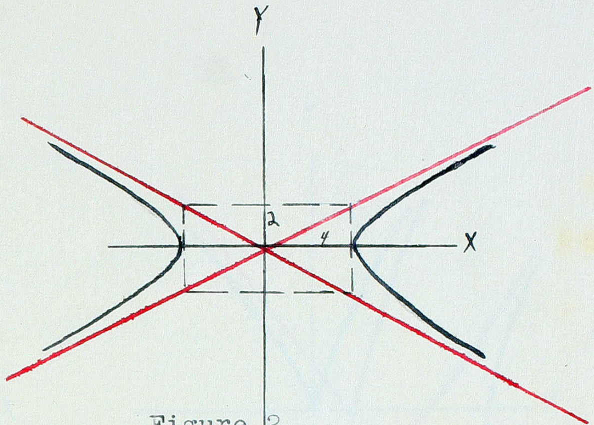

Figure 2.

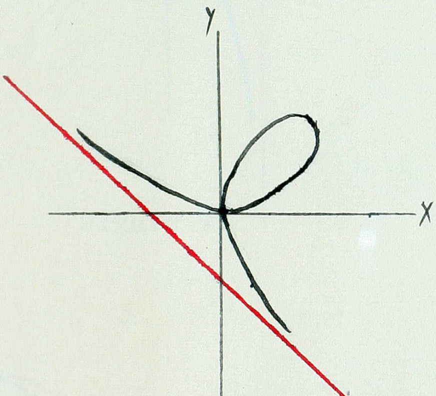

Figure 3.

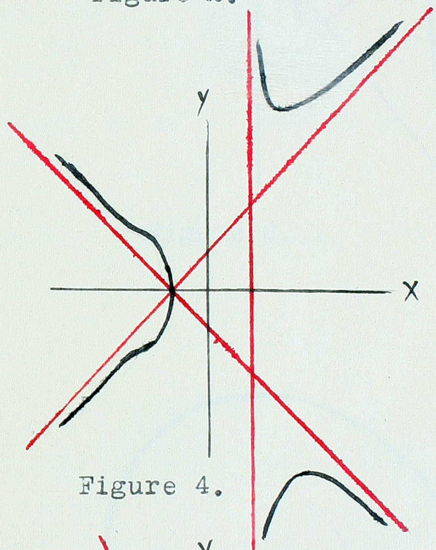

Figure 4.

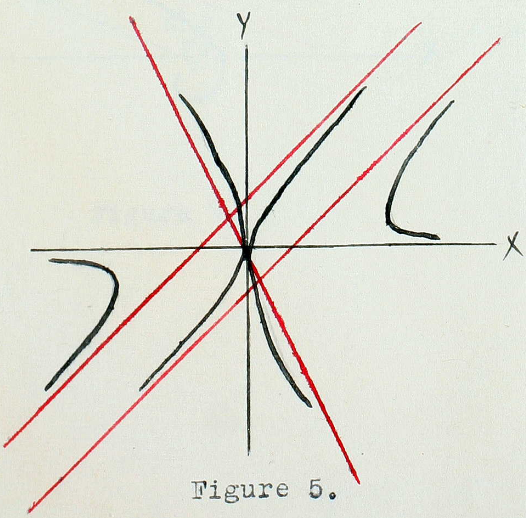

Figure 5.

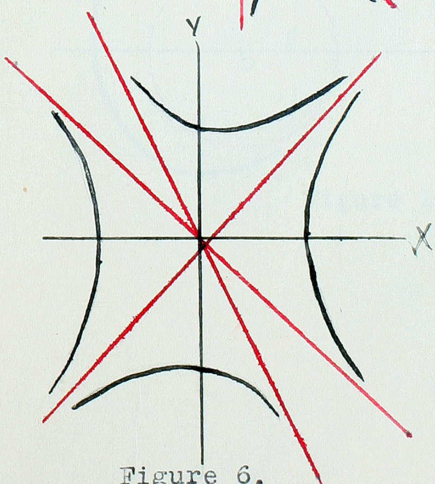

Figure 6.

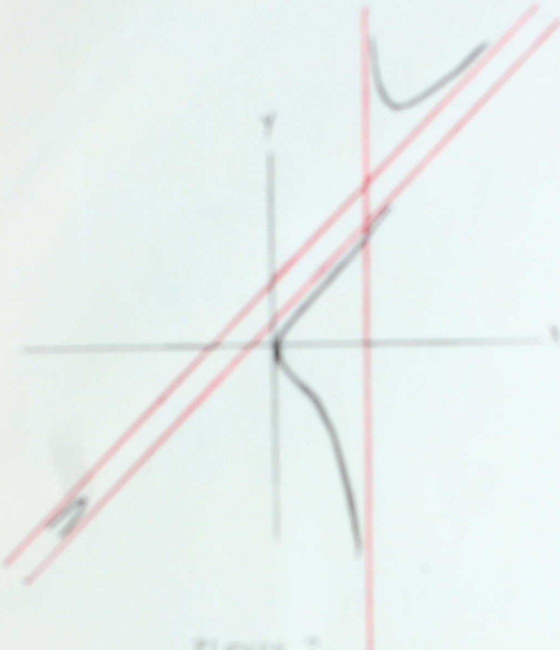

Figure 7.

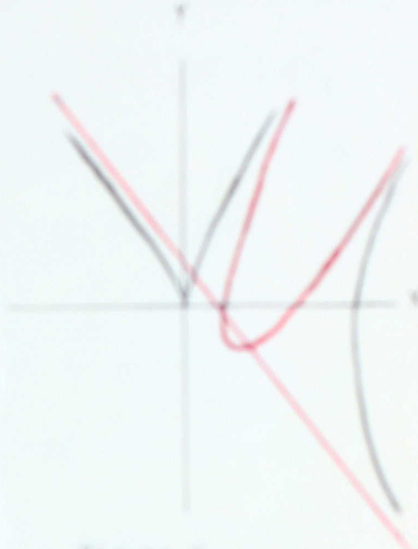

Figure 8.

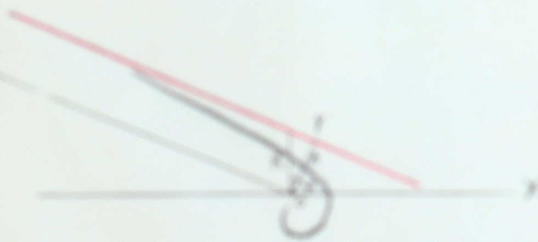

Figure 9.

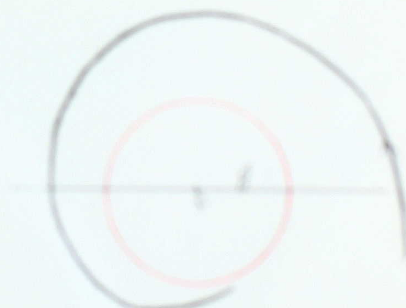

Figure 10

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