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Improving Shewhart Control Chart Performance In The Presence Of Measurement Error Using Multiple Measurements And Two-stage Sampling

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IMPROVING SHEWHART CONTROL CHART PERFORMANCE IN THE PRESENCE OF MEASUREMENT ERROR USING MULTIPLE MEASUREMENTS AND TWO-STAGE SAMPLING

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The usual Shewhart control chart efficiently detects large shifts in the mean of a quality characteristic and has been extensively studied in the literature. Most proposed alternatives to the Shewhart chart aim to improve either the signal performance for smaller mean shifts or reduce the sampling effort required to detect a larger shift. Measurement error has been shown in the literature to result in reduced power to detect process shifts. The combination of multiple measurements and two-stage sampling is considered here as a strategy for both regaining power lost due to measurement error and specifically tuning the charts for shifts of a particular size. It is shown that both the average total sample size and the average run length are improved relative to the standard Shewhart control chart under the same measurement error conditions. Chart designs are recommended to achieve particular control objectives.

Keywords: Control Chart, Average Run Length, Measurement Error, Variable Sampling

INTRODUCTION

The Shewhart control chart as presented in, e.g., Shewhart (1939) remains as perhaps the most widely employed and studied tool in statistical process control. In the simplest case, the Shewhart chart operates as a sequence of presumably independent tests of hypothesis concerning some parameter of interest. Generally, successive samples are obtained at intervals and checked against one or more control limits. Typically, in the so-called standards given case, the basic Shewhart chart for the process mean employs control limits as follows:

$$\mu \pm z_{\alpha/2} \, {}^{\sigma}\!/_{\sqrt{n}}.$$

In the preceding formula, μ is the assumed known (or desired) process mean, $z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution, σ is the assumed known (desired) process standard deviation, *n* is the sample size, and α is the desired type I error rate. The upper and lower values resulting from the preceding are generally referred to as the upper and lower control limits, or UCL and LCL, respectively.

The chart is said to produce a signal when any plotted sample mean exceeds the control limits, effectively a rejection of the null hypothesis that the process mean is equal to the in-control value μ. Many proposed modifications to this basic construction involve varying the sample size, sampling interval, placement of the control limits, number of control limits, and application of myriad alternate rejection rules.

Most modern representations of the Shewhart chart divide chart operation into two distinct phases. In phase I, either a sequence of *m* subgroups of size *n* is obtained in order to establish initial process control, or process parameters are assumed known. Phase II operation is sustained process surveillance using the process parameters established during phase I. Phase II operation under the parameters known (or standards given) case is considered here.

CHART PERFORMANCE MEASURES

The relative performance of proposed control chart schemes is generally evaluated using the average run length (ARL), or its equivalents, the signal probability or operating characteristic curves. The average run length is generally taken as the number of sampling intervals required to produce a signal. In the parameters given case, it is assumed that successive samples are independent, in which case the run length distribution is geometric with mean equal to the reciprocal of the signal probability. Therefore, chart performance is completely characterized by the mean, or average run length. Charts are typically evaluated in terms of the ARL performance associated with a sustained shift in the controlled parameter that it is desired to detect relatively quickly.

For a fair comparison, competing charts should at once possess both the same average total sample size and the same in-control average run length. A signal is generated on the Shewhart means chart if the calculated value of the sample mean falls beyond either the upper or lower control limit. If it is assumed that the quality characteristic has a normal distribution, and that at some time *t* the mean of the process shifts to some value $\mu' = \mu + \delta\sigma$, then the probability of a signal on the Shewhart control chart is $p = 1 - [P(LCL < \bar{x}_t < UCL] = 1 - \Phi\left(z_{\alpha/2} - \delta\sqrt{n}\right) + \Phi\left(-z_{\alpha/2} - \delta\sqrt{n}\right)$, where Φ() is the cumulative standard normal distribution function. The average run length (ARL) is given as $1/p$. For the in-control case, the ARL is $1/\alpha$.

Many proposed alternatives to the Shewhart control chart have been motivated by the fact that, while the Shewhart chart is very effective at quickly detecting fairly large shifts in the process mean, it is relatively slow to detect small, sustained process shifts (e.g., Montgomery (2012)). The cumulative sum, or CUSUM chart introduced by Page (1954) and the exponentially weighted moving average chart, or EWMA, introduced in Roberts (1959) are two very popular alternative charts for detecting smaller process mean shifts.

Significant work has also been done in the area of modifications to the Shewhart chart, usually focusing on either the addition of additional rejection or "runs" rules as introduced in Western Electric (1956) or adjustments to the sampling interval, as in, e.g., Reynolds et al (1988) and Pignatiello (1991), or adjustments to both the sampling interval and sample size as in Prabhu et al (1994). What is contemplated here is a variation of the theme of modifying either the sampling interval or sample size considered extensively in the literature.

TWO-STAGE SAMPLING MODEL

It is proposed here that, at each sampling interval, one sample of size n_1 be obtained followed by a contingency-dependent second sample of size n_2. The subsequent calculations are based on the assumption that a second sample may be obtained, if indicated, within a negligible

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time interval subsequent to the first sample. Three sets of control limits are used in the construction of the control chart as follows:

$$LCL_1 = \mu - z_{\alpha_1/2}\,{}^{\sigma}\!/_{\sqrt{n_1}},\; UCL_1 = \mu + z_{\alpha_1/2}\,{}^{\sigma}\!/_{\sqrt{n_1}},$$
$$LCL_2 = \mu - z_{\alpha_2/2}\,{}^{\sigma}\!/_{\sqrt{n_1}},\; UCL_2 = \mu + z_{\alpha_2/2}\,{}^{\sigma}\!/_{\sqrt{n_1}},$$
$$LCL_3 = \mu - z_{\alpha_3/2}\,{}^{\sigma}\!/_{\sqrt{n_2}},\; UCL_3 = \mu + z_{\alpha_3/2}\,{}^{\sigma}\!/_{\sqrt{n_2}},$$

where $\alpha_1 > \alpha_2$.

The operation of the proposed chart at each sampling interval is as follows:

1) The first sample mean ($\bar{x}_{t1}$) falls between LCL_1 and UCL_1, no signal is generated, and monitoring for the sampling interval is completed.
2) The sample mean ($\bar{x}_{t1}$) falls outside either LCL_1 or UCL_1, but between LCL_2 and UCL_2, after which a second sample of size n_2 is obtained.
3) The sample mean ($\bar{x}_{t1}$) falls beyond either LCL_2 or UCL_2, producing a signal.
4) The second sample mean ($\bar{x}_{t2}$) falls outside either LCL_3 or UCL_3, producing a signal, or between LCL_3 and UCL_3, completing monitoring for the sampling interval.

The probability of a signal on the two-stage control chart is calculated as follows:

$$P(Signal) = 1 - P(LCL_2 < \bar{x}_{t1} < UCL_2) + [P(LCL_2 < \bar{x}_{t1} < LCL_1) + P(UCL_1 < \bar{x}_{t1} < UCL_2)] \times [1 - P(LCL_3 < \bar{x}_{t2} < UCL_3)]$$

$$= \left[1 - \Phi\left(z_{\alpha_2/2} - \delta\sqrt{n_1}\right) + \Phi\left(-z_{\alpha_2/2} - \delta\sqrt{n_1}\right)\right] + \left[\Phi\left(z_{\alpha_2/2} - \delta\sqrt{n_1}\right) - \Phi\left(-z_{\alpha_2/2} - \delta\sqrt{n_1}\right) - \Phi\left(z_{\alpha_1/2} - \delta\sqrt{n_1}\right) + \Phi\left(-z_{\alpha_1/2} - \delta\sqrt{n_1}\right)\right] \times \left[1 - \Phi\left(z_{\alpha_3/2} - \delta\sqrt{n_2}\right) + \Phi\left(-z_{\alpha_3/2} - \delta\sqrt{n_2}\right)\right].$$

For the in-control case, $\delta = 0$ and $P(Signal) = \alpha_2 + (\alpha_1 - \alpha_2)\alpha_3$. To facilitate direct comparisons of the proposed chart with the Shewhart $\bar{x}$ chart, it is required that the in-control ARL for the proposed scheme be equal to the in-control Shewhart ARL. This is equivalent to the type I error rate for the proposed chart being equal to α, or $\alpha_2 + (\alpha_1 - \alpha_2)\alpha_3 = \alpha$. It is additionally required for a fair comparison that the average in-control sample size for the two-stage control chart be less than or equal to n, the fixed and constant sample size for the Shewhart control chart. This is accomplished through satisfying the inequality $n_1 + n_2(\alpha_1 - \alpha_2) \le n$. Treating n_1 as a chart design parameter, n_2 must be chosen to satisfy the inequality $n_2 \le (n - n_1)/(\alpha_1 - \alpha_2)$. Re-expressing α_2 as $c\alpha$, where $0<c<1$, we have $c\alpha + (\alpha_1 - c\alpha)\alpha_3 = \alpha$, which yields $\alpha_3 = (\alpha - c\alpha)/(\alpha_1 - c\alpha)$.

RESULTS

Using the constraints indicated in the previous section, values of n_1, α_1, and c were found numerically to satisfy various criteria. First, two-stage charts were designed to minimize the ARL (equivalently average time to signal) for several values of δ. Some representative results are provided in the table below.

Table 1. Average Run Length Comparisons for the Shewhart Chart versus Several Two-Stage Designs. Highlighted values indicate the shift size for which the design is optimized.

δ	Shewhart, n=10, α=.0027	n_1=1, α_1=.028, c=0	n_1=3, α_1=.085, c=0	n_1=5, α_1=.115, c=0	n_1=7, α_1=.107, c=.03	n_1=8, α_1=.109, c=.13	n_1=9, α_1=.09, c=.4
0	370.40	370.40	370.40	370.40	370.40	370.40	370.40
0.1	244.14	62.97	102.99	145.35	180.41	212.83	237.13
0.2	109.97	33.00	26.18	38.92	53.93	73.92	95.50
0.3	49.61	28.45	11.01	13.40	18.12	26.11	37.06
0.4	24.17	24.48	6.74	6.14	7.55	10.69	15.78
0.5	12.83	20.70	5.03	3.62	3.92	5.20	7.62
0.6	7.40	17.34	4.00	2.57	2.47	2.99	4.20
0.7	4.63	14.48	3.26	2.03	1.81	2.00	2.63
0.8	3.13	12.12	2.71	1.71	1.48	1.52	1.85
0.9	2.28	10.18	2.30	1.50	1.29	1.28	1.44
1.0	1.77	8.60	1.98	1.34	1.18	1.15	1.22
1.5	1.04	4.12	1.24	1.04	1.11	1.00	1.00
2.0	1.00	2.37	1.04	1.00	1.06	1.00	1.00

The results in the previous table can be somewhat misleading, insofar as the improved ARL performance for many of the shift sizes comes as a result of an increased sample size, i.e., the average sample size for the two-stage chart varies with the shift size. In practice, this is probably of little concern; however, the following table presents two-stage chart designs compared with the Shewhart chart on the basis of the average number of observations required to produce a signal at a given shift size.

Table 2. Comparisons of Average Total Observations to Signal for the Shewhart Chart versus Several Two-Stage Designs. Highlighted values indicate the shift size for which the design is optimized.

δ	Shewhart, n=10, α=.0027	n_1=1, α_1=.028, c=0	n_1=1, α_1=.117, c=0	n_1=1, α_1=.198, c=0	n_1=1, α_1=.420, c=0	n_1=9, α_1=.096, c=.67	n_1=9, α_1=.164, c=.82
0	3703.98	3693.63	3663.98	3670.65	3637.31	3682.47	3693.14
0.1	2441.38	644.84	1035.40	1347.32	1880.32	2437.24	2531.71
0.2	1099.67	363.04	281.50	382.31	659.17	1073.77	1198.70
0.3	496.10	349.48	131.64	152.66	263.01	465.03	559.28
0.4	241.71	345.48	93.32	84.15	124.57	219.04	279.14
0.5	128.25	341.70	83.71	60.25	69.55	114.45	150.50
0.6	74.02	338.34	81.63	51.71	45.14	66.30	87.39
0.7	46.34	335.48	80.89	48.94	33.43	42.26	54.39
0.8	31.34	333.12	80.34	48.11	27.60	29.34	36.12
0.9	22.79	331.18	79.86	47.76	24.72	21.92	25.52
1.0	17.72	329.60	79.44	47.51	23.37	17.42	19.12
1.5	10.42	325.12	78.11	46.70	22.30	9.85	9.62
2.0	10.00	323.36	77.50	46.30	22.13	9.02	9.01

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CONCLUSIONS

With a fairly simple modification, it is possible to dramatically improve upon the basic Shewhart chart with respect to either average run length or average total sampling effort to produce a signal, and frequently both at once, for certain specified ranges of process mean shifts. No comparison is made here, although a cursory investigation indicates that the two-stage chart is a capable rival for the popular CUSUM and EWMA control charts. Future investigations should consider direct comparisons between the two-stage procedure discussed here and other, more established procedures. Analyses of the performance of the two-stage chart under other conditions such as estimated parameters, non-normality of the quality characteristic, and measurement error should also be added.

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