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Continued fraction: A research on Markov triples

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INTRODUCTION

Being one of the many studies in Number Theory, continued fractions is a study of math that offer useful means of expressing numbers and functions. Mathematicians in early ages used algorithms and methods to express numbers. Many of these expressions turned into the study of continued fractions.

Since the start of the 20th century, mathematicians have been able to develop continued fractions in other areas due to the prevalence of personal computer. Computer algorithms started to be used for computing rational approximations to real numbers and solving complex equations.

GENERAL FORM OF CONTINUED FRACTION

$$r = 0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}$$

where a_i and b_i are either rational numbers, real numbers, or complex numbers. If $b_i = 1$ for all i, then the expression is called a simple continued fraction. If the expression contains finitely many terms, then it is called a finite continued fraction; otherwise, it is called an infinite continued fraction.

MARKOV TRIPLES

As of our research, we chose Markov Triples as the concentration. A Markov triple (x, y, z) consists of three positive integers such that:

$$x^2 + y^2 + z^2 = 3xyz$$

A Markov number is an integer that belongs to at least one Markov triple. Two Markov triples are said to be adjacent to each other if two of their Markov numbers are the same. For example, (5, 1, 2) is a Markov triple and its neighbors are (1, 1, 2), (5, 29, 2), (5, 1, 13).

STUDY PROCESS

Our study of Markov Triple aimed at a simple level since the main researcher, Daheng Chen who was still a high school student, did not have enough professional knowledge in math. Professor Hongbiao Zeng left three questions as the mainstream of the whole study.

Professor Zeng came up with the following three questions regarding Markov triples:

1 VERIFY (5, 1, 2) IS A MARKOV TRIPLE

$$5^2 + 1^2 + 2^2 = 25 + 1 + 4 = 30 \qquad 3xyz = 3 \cdot 5 \cdot 1 \cdot 2 = 30$$

After verification, we found that (5, 1, 2) is a Markov triple according to Markov equation.
This question is an intro to the consecutive problems, it inspired us the methods to find more Markov numbers and their connections with other numbers.

2 FIND CONNECTION WITH FIBONACCI SEQUENCE

As we were computing some Markov numbers, we found out some interesting patterns. Comparing the first few numbers of Fibonacci sequence and Markov equation:

Fibonacci sequence: {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597}

Markov numbers: {1, 2, 5, 13, 29, 34, 89, 169, 194, 233, 433, 610, 985, 1325, 1597}

3 FIND CONNECTION WITH CONTINUED FRACTIONS

Our final goal is to start with a Markov triple and iteratively construct Markov triples by constructing the neighbors at each step and try to find certain periodic continued fractions appear as roots of quadratic forms associated to Markov triples. The secondary goal is to see what the roots are for the chain of Markov triples. However, we finally failed in finding any pattern in periods of continued fractions as roots.

OUTCOMES

When solving the second question, we saw the clear relationship between Fibonacci sequence and Markov numbers: all odd indices of Fibonacci sequence are a subset of Markov numbers.
We did not reach our ultimate research goal at this time; however, both of us gained precious experience in this area where we were not familiar with. The processes of finding interesting relationship between numbers invoked our passion to delve more into number theory, continued fractions especially.
In the future, we will continue in this research and hopefully to have exciting findings.

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